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SMALL DEFORMATIONS SUPERPOSED ON LARGE DEFORMATIONS OF AN ELASTIC–PLASTIC MATERIAL

J. CASEY
Department of Mechanical Engineering, University of Houston, Houston, TX 77004, U.S.A.

and

P. M. NAGHDI
Department of Mechanical Engineering, University of California, Berkeley, CA 94720, U.S.A.

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Abstract—In the context of a purely mechanical rate-type theory of plasticity, and utilizing a strain space formulation, an infinitesimal theory is developed for motions superposed on any given motion of an elastic–plastic material. The given motion may involve all forms of elastic–plastic deformation, including both loading and unloading, and in addition, the loading conditions are allowed to change due to the superposed motion. The infinitesimal theory is properly invariant under arbitrary finite superposed rigid motions.

1. INTRODUCTION

A general thermodynamical theory of finitely deforming elastic–plastic materials was presented by Green and Naghdi [1, 2] and has been utilized in a large number of subsequent developments. The basic equations of the theory in [1, 2] are nonlinear ordinary and partial differential equations. Due to the lack of general mathematical techniques for dealing with such equations, it is extremely difficult to obtain exact solutions to all but the simplest of initial boundary-value problems. To treat more difficult problems, one must resort to approximate methods.

The theory of small deformations superposed on an arbitrary known deformation is based on a systematic local linear approximation to the nonlinear theory. This type of approximation to a general theory has been found to provide in special cases a reasonable approximation to the observed behavior of materials. Indeed, the classical linear theories of elasticity and plasticity, which themselves may be regarded as theories of small deformations superposed on a zero deformation, have met with much success in their predictions of the behavior of metals even under moderately large deformations.

In the context of nonlinear elasticity, the general theory of small deformations superposed on a large deformation originated in the 1950s with the work of Green *et al.* [3]. An account of this theory, with several applications, is contained in [4]. A somewhat different treatment of the theory was given by Toupin and Bernstein [5]. A presentation of the subject together with additional references is to be found in [6].

A shortcoming of the above formulations of the theory of small deformations superposed on a large deformation of an elastic material is that requirements of invariance under arbitrary superposed rigid body motions are violated. Recently, however, Casey and Naghdi [7] have introduced a method by the use of which the theory may be put into properly invariant form. The method in [7] consists of removing from every motion of a body the translation and rotation at any one particle while maintaining at all particles the finite strain of the original motions. When the motions derived in this manner are used instead of the original ones in constructing an approximate theory, the results are properly invariant under arbitrary finite superposed rigid motions.

Leaving aside the problem of invariance for the moment, we note that, compared to the theory of small deformations superposed on a large deformation of an elastic material, the corresponding theory for an elastic–plastic material involves additional difficulties. These difficulties are mainly due to the possible occurrence of changes in the condition of plastic loading as a result of the superposed deformation. In the context of the theory of Green and Naghdi [1,2], Shack [8] constructed a special theory of small deformations superposed on a large deformation. In particular, Shack examined how elastic–plastic materials respond when at some instant t_0 the velocity field associated with any given three-dimensional elastic–plastic motion is perturbed in such a manner that for a sufficiently short period following t_0 no unloading of the material occurs. The possibility of this type of phenomenon was first recognized by Shanley [9] in his analysis of the bending of a uniformly compressed column, and was further studied in the case of plane strain perturbations on dominating uniform compressive plastic flows by Goodier [10] and by Ramsey [11].

We recall that the traditional formulation of plasticity theory employs yield surfaces in stress space, together with loading criteria which involve the time rate of stress. Such criteria are known to lead to reliable results in the work-hardening range of elastic–plastic materials. For these reasons, a stress (as well as temperature) space formulation of the nonlinear theory was adopted in [1, 2] and hence in [8]. However, the loading criteria of the stress space formulation of plasticity are not valid in a region such as that corresponding to the neighborhood of the maximum point, and the falling portion, of the engineering stress versus engineering strain curve for uniaxial tension (i.e. in the vicinity of ultimate strength, and during necking, of the material). This was observed by Naghdi and Trapp [12], who also proposed an alternative strain space formulation of plasticity† which is free from the shortcomings of the stress space formulation. Furthermore, the strain space formulation was shown in [12] to have the additional advantage that the loading criteria for perfectly plastic materials are the same as for work-hardening materials, whereas in the stress space formulation the loading criteria for these two classes of materials are different and a separate treatment is required for perfectly plastic materials. More recently, Casey and Naghdi [14] have shown that if the loading criteria of the strain space formulation are adopted as primary, then the induced behavior in stress space can be used to define in a natural way three distinct types of strain-hardening response, namely hardening, softening and perfectly plastic behavior. In the context of the development of [14], it is evident that the usual stress space formulation includes only hardening behavior and is incapable of treating softening and perfectly plastic behavior.

The strain space formulation [12] was elaborated upon further in [15], which also contains a discussion of a physically plausible assumption originally introduced in a strain space setting by Naghdi and Trapp [16]. This assumption [16], which is concerned with nonnegativity of work in a closed cycle of homogeneous deformation, implies certain restrictions on the general constitutive equations of elastic–plastic materials and was further examined in [17].

In the context of the purely mechanical theory developed in [12, 14, 16], and applying the scheme of [7], we introduce in this paper a properly invariant infinitesimal theory of motions superposed on any given motion of an elastic–plastic body. This theory includes as a special case an invariant infinitesimal theory of elastic–plastic materials. Also, upon specialization to an elastic material, it reduces to the theory of small deformations superposed on a large deformation presented in [7]. If we consider the isothermal case of Shack's thermodynamical treatment [8] of the problem we find that the present development differs from his in the following ways:

(i) Firstly, we allow the given motion to involve all forms of elastic–plastic deformation including both loading and unloading; and, furthermore, we allow these conditions to change as a result of the superposed motion. In contrast, adopting the idea of Shanley, Shack restricts his attention to only dominating plastic flows.

(ii) Secondly, we utilize a strain space rather than stress space formulation of plasticity.

(iii) Thirdly, in our development of an infinitesimal theory of motions superposed on a given motion, we consider two independent motions ${}_1\chi$ and ${}_2\chi$ of an elastic–plastic material, ${}_1\chi$ being identified with the given motion. If we were to neglect the question of invariance, we would then construct our approximate theory by letting the gradient of the displacement field of ${}_2\chi$ relative to ${}_1\chi$ approach zero at each instant of time, assuming in addition that the time derivative of this gradient, as well as other quantities associated with the elastic–plastic properties of the material, were small. This corresponds to the practical

†The development in [12] is carried out within a purely mechanical framework which can readily be interpreted in terms of the isothermal case of the thermodynamical theory [1, 2]. The corresponding thermodynamical theory is discussed in [13].

situation in which (apart from an infinitesimal rigid motion) the two motions ${}_1\chi$ and ${}_2\chi$ differ during their entire history only by displacements and velocities of $O(\epsilon)$. In contrast, Shack considers the given motion to undergo at some time t_0 a perturbation in its velocity with no change in position. He confines his analysis to a sufficiently short interval following t_0 during which the given motion changes at most by terms of $O(\epsilon^2)$ from its current values in the unperturbed state. We may obtain Shack's special perturbations by considering our ${}_2\chi$ to coincide with ${}_1\chi$ up to and including time t_0 and comparing our results to his as $t \to t_0$ from above.

(iv) Fourthly and lastly, our theory is properly invariant under arbitrary finite rigid motions superposed on either ${}_1\chi$ or ${}_2\chi$ or both.

Before closing this section, we outline the contents of the remainder of the paper. Section 2 contains the basic field equations of the purely mechanical theory of a finitely deforming body. Again in the context of the purely mechanical theory, in Section 3 we summarize the constitutive theory of an elastic–plastic material of the rate-type. In Section 4 we utilize the scheme proposed by Casey and Naghdi [7] as a means of constructing invariant infinitesimal theories. Finally, in Section 5 the equations governing small motions superposed on an arbitrary known motion of an elastic–plastic material are derived.

2. GENERAL BACKGROUND. PRELIMINARIES AND NOTATION

Consider a body $\mathscr{B}$ which, in a fixed reference configuration ${}_0\kappa$ occupies a region ${}_0\mathscr{R}$, with boundary $\partial_0\mathscr{R}$, in a three-dimensional Euclidean space $\mathscr{E}^3$. Choosing a fixed origin $\mathscr{O}$ in $\mathscr{E}^3$, we identify each particle X of $\mathscr{B}$ by the position vector $\mathbf{X}$ of the place it occupies in ${}_0\mathscr{R}$. A motion of $\mathscr{B}$ is a mapping χ which assigns a position vector $\mathbf{x} = \chi(\mathbf{X}, t)$ to each particle X at each instant t of time $(-\infty < t < \infty)$. In what follows, we need to consider three separate motions of the body; and, in line with the notation of [7], introduce

$$ {}_\alpha\mathbf{x} = {}_\alpha\chi(\mathbf{X}, t), \tag{2.1} $$

where α takes on values 0, 1, 2 and ${}_0\mathbf{x} = \mathbf{X}$. The motion ${}_0\mathbf{x} = \mathbf{X} = {}_0\chi(\mathbf{X}, t)$ in which $\mathbf{X}$ remains at $\mathbf{X}$ for all t is called the *identity motion*. We note that ${}_0\mathbf{x} = {}_0\kappa(X)$ and ${}_0\mathscr{R} = {}_0\kappa(\mathscr{B})$. In our analysis of motions superposed on a given motion, ${}_1\chi$ will represent the given motion and ${}_2\chi$ a motion that is close to ${}_1\chi$ in a sense to be made precise. The image of ${}_0\mathscr{R}$ in the motion ${}_\alpha\chi$ will be denoted by ${}_\alpha\mathscr{R} = {}_\alpha\chi({}_0\mathscr{R}, t)$. We assume that at each fixed t, the mapping of $\mathscr{R}_0$ into $\mathscr{R}$ by (2.1) possesses a smooth inverse denoted by ${}_\alpha\chi^{-1}$. Under these assumptions, for each α, ${}_\alpha\mathscr{R}$ is also a region with boundary $\partial_\alpha\mathscr{R} = {}_\alpha\chi(\partial_0\mathscr{R}, t)$. The current configuration of $\mathscr{B}$ at each fixed t in the motion ${}_\alpha\chi$ is the mapping ${}_\alpha\kappa$ of $\mathscr{B}$ into $\mathscr{E}^3$ given by ${}_\alpha\kappa = {}_\alpha\chi o_0\kappa$, where o signifies the composition of mappings. For any subset (or part) $\mathscr{S} \subseteq \mathscr{B}$ of the body, we write ${}_0\mathscr{P} = {}_0\kappa(\mathscr{S})$, ${}_\alpha\mathscr{P} = {}_\alpha\chi({}_0\mathscr{P}, t)$ and $\partial_\alpha\mathscr{P} = {}_\alpha\chi(\partial_0\mathscr{P}, t)$, where $\partial_0\mathscr{P}$ is the boundary of the region ${}_0\mathscr{P}$ and $\partial_\alpha\mathscr{P}$ that of ${}_\alpha\mathscr{P}$.

Before continuing with the kinematics, we mention some mathematical terminology that will be needed in what follows. Any linear mapping from V^3, the three-dimensional translation vector space associated with the point space $\mathscr{E}^3$, into V^3 will be called a second order tensor. The trace and determinant functions are denoted respectively by tr and det. The transpose of a second order tensor $\mathbf{A}$ will be denoted by $\mathbf{A}^T$, while the inverse of $\mathbf{A}$ if it exists will be denoted by $\mathbf{A}^{-1}$. The usual inner product on V^3 is written $\mathbf{a} \cdot \mathbf{b}$ for any two vectors $\mathbf{a}, \mathbf{b} \in V^3$ and the (induced) norm, or magnitude of $\mathbf{a}$ is given by $\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$. The set of second order tensors can be provided with an inner product $\mathbf{A} \cdot \mathbf{B} = \mathrm{tr}(\mathbf{A}^T\mathbf{B})$ for any two second order tensors $\mathbf{A}$ and $\mathbf{B}$, and a norm $\|\mathbf{A}\| = \sqrt{\mathbf{A} \cdot \mathbf{A}}$. The tensor product $\mathbf{a} \otimes \mathbf{b}$ of any two vectors $\mathbf{a}, \mathbf{b} \in V^3$ is the second order tensor defined by $(\mathbf{a} \otimes \mathbf{b})\mathbf{u} = \mathbf{b} \cdot \mathbf{u}\, \mathbf{a}$ for every vector $\mathbf{u}$. We recall the formulae $\mathrm{tr}(\mathbf{a} \otimes \mathbf{b}) = \mathbf{a} \cdot \mathbf{b}$, $(\mathbf{a} \otimes \mathbf{b})^T = \mathbf{b} \otimes \mathbf{a}$ and $(\mathbf{a} \otimes \mathbf{b})(\mathbf{c} \otimes \mathbf{d}) = \mathbf{b} \cdot \mathbf{c}\, \mathbf{a} \otimes \mathbf{d} = (\mathbf{a} \otimes \mathbf{c})(\mathbf{b} \otimes \mathbf{d})$ which hold for all vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ in V^3. The convention of summation over a repeated index will be employed, except for a repeated index which appears to the left of a central character.

In order to express certain expressions in component form, it is convenient to employ two fixed right-handed orthonormal bases $\{\mathbf{e}_A\}$ and $\{\mathbf{e}_i\}$ in V^3, the former basis being used for vector fields defined on the region ${}_0\mathscr{R}$ and the latter for vector fields defined on other regions. Thus, for example, we write $\mathbf{X} = X_A\mathbf{e}_A$ and ${}_\alpha\mathbf{x} = {}_\alpha x_i\mathbf{e}_i$ $(\alpha = 1, 2)$. Furthermore, a second order tensor A may be represented by $A_{ij}e_i \otimes e_j$, $A_{iM}e_i \otimes e_M$ or $A_{MN}e_M \otimes e_N$ as appropriate, where $A_{ij} = e_i \cdot Ae_j = A \cdot (e_i \otimes e_j)$, etc. The second order identity tensor is denoted by I and may be represented by $e_i \otimes e_i$, etc. Any linear mapping from the set of second order tensors into itself is a fourth order tensor. In particular, the tensor product $a \otimes b \otimes c \otimes d$ of any four vectors $a, b, c, d \in V^3$ is a fourth order tensor. It is useful to define an inner product of the fourth order tensor $a \otimes b \otimes c \otimes d$ and the second order tensor $u \otimes v$, $u, v \in V^3$, by $(a \otimes b \otimes c \otimes d)[u \otimes v] = c \cdot u\, d \cdot v\, a \otimes b$, which yields a second order tensor. Any fourth order tensor $\mathscr{A}$ may be represented as $\mathscr{A} = \mathscr{A}_{ijkl} e_i \otimes e_j \otimes e_k \otimes e_l = \mathscr{A}_{KLMN} e_K \otimes e_L \otimes e_M \otimes e_N = \mathscr{A}_{ijKL} e_i \otimes e_j \otimes e_K \otimes e_L$, etc. where, for example, $\mathscr{A}_{ijkl} = e_i \cdot \mathscr{A}[e_k \otimes e_l]e_j = (e_i \otimes e_j) \cdot \mathscr{A}[e_k \otimes e_l]$. The transpose $\mathscr{A}^T$ of a fourth order tensor $\mathscr{A}$ is that unique fourth order tensor with the property that $B \cdot \mathscr{A}[A] = A \cdot \mathscr{A}^T[B]$ for all second order tensors A, B. Clearly, $\mathscr{A}^T = (\mathscr{A}_{ijkl} e_i \otimes e_j \otimes e_k \otimes e_l)^T = \mathscr{A}_{klij} e_i \otimes e_j \otimes e_k \otimes e_l$.

We also need to consider third order tensors, which may be viewed as linear mappings from the set of second order tensors into V^3. Let $a, b, c, u, v \in V^3$ and define a product $(a \otimes b \otimes c)[u \otimes v] = b \cdot u\, c \cdot v\, a$. Then $a \otimes b \otimes c$ is a third order tensor. We also introduce here the products $(a \otimes b \otimes c)(u \otimes v) = c \cdot u\, a \otimes b \otimes v$ and $(a \otimes b \otimes c)u = c \cdot u\, a \otimes b$.

Having disposed of the foregoing notational preliminaries, we return to further consideration of kinematics. The deformation gradient ${}_\alpha F$, associated with the motion ${}_\alpha\chi$, relative to X is defined by

$$ {}_\alpha F = \frac{\partial {}_\alpha\chi}{\partial X}(X, t), \quad {}_\alpha J = \det({}_\alpha F) > 0. \tag{2.2}$$

Being invertible, ${}_\alpha F$ possesses a unique polar decomposition in the form

$$ {}_\alpha F = {}_\alpha R\, {}_\alpha U, \tag{2.3}$$

where the (local) rotation ${}_\alpha R$ is a proper orthogonal second order tensor and the right stretch ${}_\alpha U$ is a symmetric positive-definite second order tensor. Also, the right Cauchy–Green measure of deformation ${}_\alpha C$ and the Lagrangian finite strain tensor ${}_\alpha E$ are given by

$$ {}_\alpha C = {}_\alpha U^2 = {}_\alpha F^T {}_\alpha F, \quad {}_\alpha E = \tfrac{1}{2}({}_\alpha C - I). \tag{2.4}$$

The relative displacement field associated with the motion ${}_\alpha X$ is the mapping ${}_\alpha\chi - {}_0\chi$ with the values

$$ {}_\alpha u = ({}_\alpha\chi - {}_0\chi)(X, t) = {}_\alpha x - X \tag{2.5}$$

and its gradient relative to X, namely

$$ {}_\alpha G = \frac{\partial({}_\alpha\chi - {}_0\chi)}{\partial X}(X, t) = {}_\alpha F - I, \tag{2.6}$$

is called the displacement gradient.

A motion ${}_\alpha\chi^+$ of $\mathscr{B}$ is said to differ from ${}_\alpha\chi$ by a superposed rigid motion if and only if

$$ {}_\alpha\chi^+(X, {}_\alpha t^+) = {}_\alpha Q(t)\, {}_\alpha\chi(X, t) + {}_\alpha a(t), \quad {}_\alpha t^+ = t + {}_\alpha a \tag{2.7}$$

for some proper orthogonal second order tensor-valued function ${}_\alpha Q(t)$ of time, some vector-valued function ${}_\alpha a(t)$ of time, and some real constant ${}_\alpha a$. The configuration of $\mathscr{B}$, at time ${}_\alpha t^+$, in the motion ${}_\alpha\chi^+$ is ${}_\alpha\kappa^+ = {}_\alpha\chi^+ o_0\kappa$. The class of rigid motions of $\mathscr{B}$ consists of those motions ${}_0\chi^+$ which differ from the identity motion ${}_0\chi$ by a rigid motion, being given by (2.7) with $\alpha = 0$.

It was observed in [7] that the statement "differs by a rigid motion" is an equivalence relation on the set $\mathscr{M}$ of all motions of $\mathscr{B}$. This allows $\mathscr{M}$ to be partitioned into disjoint subsets (equivalence classes) each of which comprises all motions of $\mathscr{B}$ and only those, which differ from one another by a rigid motion. Thus, each equivalence class comprises those motions of $\mathscr{B}$ which are regarded as mechanically indistinguishable. The equivalence class which contains all motions that differ from a given motion χ by a rigid motion is denoted by $K(\chi)$. For example, the equivalence class $K({}_0\chi)$ contains the entire set of rigid motions of $\mathscr{B}$. We also recall from [7] that an equivalence class is determined by any one of its members. If, instead of a motion χ, we begin with a motion $\boldsymbol{\theta}$ and place all the members of $\mathscr{M}$ that are equivalent to $\boldsymbol{\theta}$ in a class $K(\boldsymbol{\theta})$, we find that $K(\boldsymbol{\theta}) = K(\chi)$. Any member of an equivalence class is called a *representative* of the class.

We may regard (2.7) as defining a function ω taking $\mathscr{M}$ into $\mathscr{M}$ such that for fixed values of ${}_\alpha\boldsymbol{Q}, {}_\alpha\boldsymbol{a}$ and ${}_\alpha a$ in (2.7)

$${}_\alpha\chi^{+} = \omega({}_\alpha\chi). \tag{2.8}$$

From the deformation gradient ${}_\alpha\boldsymbol{F}^{+} = (\partial_\alpha\chi^{+}/\partial\boldsymbol{X})(\boldsymbol{X}, {}_\alpha t^{+})$ of ${}_\alpha\chi^{+}$ in $(2.7)_1$ and (2.2), we readily obtain

$${}_\alpha\boldsymbol{F}^{+} = {}_\alpha\boldsymbol{Q}(t)_\alpha\boldsymbol{F}, \quad {}_\alpha J^{+} = \det({}_\alpha\boldsymbol{F}^{+}) = {}_\alpha J > 0. \tag{2.9}$$

Then, using ${}_\alpha\boldsymbol{F}^{+}$ to define, as in (2.3) and (2.4), the tensors ${}_\alpha\boldsymbol{R}^{+}$, ${}_\alpha\boldsymbol{U}^{+}$, ${}_\alpha\boldsymbol{C}^{+}$ and ${}_\alpha\boldsymbol{E}^{+}$, it follows at once that

$${}_\alpha\boldsymbol{U}^{+} = {}_\alpha\boldsymbol{U}, \quad {}_\alpha\boldsymbol{C}^{+} = {}_\alpha\boldsymbol{C}, \quad {}_\alpha\boldsymbol{E}^{+} = {}_\alpha\boldsymbol{E}, \quad {}_\alpha\boldsymbol{R}^{+} = {}_\alpha\boldsymbol{Q}(t)_\alpha\boldsymbol{R}. \tag{2.10}$$

The relative displacement field associated with ${}_\alpha\chi^{+}$ is ${}_\alpha\boldsymbol{u}^{+} = ({}_\alpha\chi^{+} - {}_0\chi)(\boldsymbol{X}, {}_\alpha t^{+})$ and its gradient is ${}_\alpha\boldsymbol{G}^{+} = {}_\alpha\boldsymbol{F}^{+} - \boldsymbol{I}$. Hence, in view of $(2.9)_1$ and (2.6)

$${}_\alpha\boldsymbol{G}^{+} = {}_\alpha\boldsymbol{Q}(t)_\alpha\boldsymbol{G} + {}_\alpha\boldsymbol{Q}(t) - \boldsymbol{I}, \tag{2.11}$$

so that ${}_\alpha\boldsymbol{G}^{+}$ is neither unaltered, nor unaltered apart from orientation† under all superposed rigid body motions of $\mathscr{B}$.

Using a superposed dot to signify material time differentiation, the particle velocity in the motion ${}_\alpha\chi$ is given by

$${}_\alpha\boldsymbol{v} = {}_\alpha\dot{\boldsymbol{x}} = \frac{\partial_\alpha\chi}{\partial t}(\boldsymbol{X}, t) \tag{2.12}$$

and the particle acceleration is ${}_\alpha\dot{\boldsymbol{v}}$. We recall the formulae

$${}_\alpha\dot{\boldsymbol{E}} = {}_\alpha\boldsymbol{F}^T{}_\alpha\boldsymbol{D}_\alpha\boldsymbol{F}, \quad {}_\alpha\boldsymbol{D} = \tfrac{1}{2}({}_\alpha\boldsymbol{L} + {}_\alpha\boldsymbol{L}^T), \quad {}_\alpha\boldsymbol{L} = {}_\alpha\dot{\boldsymbol{F}}_\alpha\boldsymbol{F}^{-1}. \tag{2.13}$$

${}_\alpha\boldsymbol{L}$ is the velocity gradient; its symmetric part ${}_\alpha\boldsymbol{D}$ is the rate of deformation tensor and ${}_\alpha\dot{\boldsymbol{E}}$ is the rate of strain tensor.

It follows from $(2.10)_{2,3}$, $(2.7)_2$, $(2.13)_1$ and $(2.9)_1$ that

$$\begin{gathered}{}_\alpha\dot{\boldsymbol{C}}^{+} = \frac{\partial({}_\alpha\boldsymbol{C}^{+})}{\partial_\alpha t^{+}}(\boldsymbol{X}, {}_\alpha t^{+}) = \frac{\partial_\alpha\boldsymbol{C}}{\partial t}(\boldsymbol{X}, t) = {}_\alpha\dot{\boldsymbol{C}},\\ {}_\alpha\dot{\boldsymbol{E}}^{+} = {}_\alpha\dot{\boldsymbol{E}},\\ {}_\alpha\boldsymbol{D}^{+} = {}_\alpha\boldsymbol{Q}(t)_\alpha\boldsymbol{D}_\alpha\boldsymbol{Q}^T(t).\end{gathered} \tag{2.14}$$

Let ${}_\alpha\rho$ be the mass density in the configuration ${}_\alpha\boldsymbol{\kappa}$, ${}_\alpha\boldsymbol{b}$ the body force field per unit mass in the configuration ${}_\alpha\boldsymbol{\kappa}$, ${}_\alpha\boldsymbol{n}$ the outward unit normal to the surface $\partial_\alpha\mathscr{P}$, ${}_\alpha\boldsymbol{t}$ the surface force vector acting on $\partial_\alpha\mathscr{P}$, and measured per unit area of $\partial_\alpha\mathscr{P}$, and ${}_\alpha\boldsymbol{T}$ the associated Cauchy stress tensor. Then, in any motion ${}_\alpha\chi$, from conservation laws for mass, linear and angular momentum follow the results

$$\begin{gathered}{}_0\rho = {}_\alpha\rho_\alpha J,\\ {}_\alpha\boldsymbol{t} = {}_\alpha\boldsymbol{T}_\alpha\boldsymbol{n}, \quad {}_\alpha\boldsymbol{T}^T = {}_\alpha\boldsymbol{T},\\ {}_\alpha\mathrm{div}\,{}_\alpha\boldsymbol{T} + {}_\alpha\rho_\alpha\boldsymbol{b} = {}_\alpha\rho_\alpha\dot{\boldsymbol{v}}.\end{gathered} \tag{2.15}$$

†See [18] for the motivation for, and the precise meaning of this terminology.

In (2.15), ${}_\alpha\mathrm{div}$ is the (right) divergence operator with respect to ${}_\alpha\boldsymbol{x}$, having a component representation

$$ {}_\alpha\mathrm{div}\,{}_\alpha\boldsymbol{T} = \frac{\partial}{\partial_\alpha x_k}({}_\alpha T_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j)\boldsymbol{e}_k = \frac{\partial_\alpha T_{ij}}{\partial_\alpha x_j}\boldsymbol{e}_i = \frac{\partial_\alpha T_{ij}}{\partial_\alpha x_m}(\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_m)[\boldsymbol{e}_k \otimes \boldsymbol{e}_k] = \frac{\partial_\alpha \boldsymbol{T}}{\partial_\alpha \boldsymbol{x}}[\boldsymbol{I}]. \quad (2.16) $$

In the motion ${}_\alpha\boldsymbol{\chi}$ the unit normal ${}_0\boldsymbol{n}$ is carried into ${}_\alpha\boldsymbol{n}$ with

$$ {}_\alpha\boldsymbol{n} = \frac{({}_\alpha\boldsymbol{F}^{-1})^T{}_0\boldsymbol{n}}{\|({}_\alpha\boldsymbol{F}^{-1})^T{}_0\boldsymbol{n}\|}, \quad {}_0\boldsymbol{n} = \frac{{}_\alpha\boldsymbol{F}^T{}_\alpha\boldsymbol{n}}{\|{}_\alpha\boldsymbol{F}^T{}_\alpha\boldsymbol{n}\|}. \quad (2.17) $$

Denoting the mass density in the configuration ${}_\alpha\boldsymbol{\kappa}^+$ by ${}_\alpha\rho^+$, applying $(2.15)_1$ to the motion ${}_\alpha\boldsymbol{\chi}^+$ as well as to ${}_\alpha\boldsymbol{\chi}$ and invoking $(2.9)_2$, we obtain

$$ {}_0\rho = {}_\alpha\rho^+\,{}_\alpha J^+, \quad {}_\alpha\rho^+ = {}_\alpha\rho. \quad (2.18) $$

It follows from (2.18) and $(2.9)_1$ that under the transformations (2.7), ${}_\alpha\boldsymbol{n}$ is carried into ${}_\alpha\boldsymbol{n}^+$, the outward unit normal to $\partial_\alpha\mathscr{P}^+$, with

$$ {}_\alpha\boldsymbol{n}^+ = {}_\alpha\boldsymbol{Q}(t){}_\alpha\boldsymbol{n}. \quad (2.19) $$

We adopt the usual assumption that the stress vector ${}_\alpha\boldsymbol{t}^+$ for the motion ${}_\alpha\boldsymbol{\chi}^+$ is related to ${}_\alpha\boldsymbol{t}$ by

$$ {}_\alpha\boldsymbol{t}^+ = {}_\alpha\boldsymbol{Q}(t){}_\alpha\boldsymbol{t} \quad (2.20) $$

and it then follows with the aid of $(2.15)_2$ and (2.19) that the Cauchy stress tensor ${}_\alpha\boldsymbol{T}^+$ in the motion ${}_\alpha\boldsymbol{\chi}^+$ is related to ${}_\alpha\boldsymbol{T}$ by

$$ {}_\alpha\boldsymbol{T}^+ = {}_\alpha\boldsymbol{Q}(t){}_\alpha\boldsymbol{T}{}_\alpha\boldsymbol{Q}^T(t). \quad (2.21) $$

The balance of linear momentum in the motion ${}_\alpha\boldsymbol{\chi}^+$ is written as

$$ {}_\alpha\mathrm{div}^+\,{}_\alpha\boldsymbol{T}^+ + {}_\alpha\rho^+{}_\alpha\boldsymbol{b}^+ = {}_\alpha\rho^+{}_\alpha\dot{\boldsymbol{v}}^+, \quad (2.22) $$

where

$$ {}_\alpha\mathrm{div}^+\,{}_\alpha\boldsymbol{T}^+ = \frac{\partial_\alpha T^+_{ij}}{\partial_\alpha x^+_j}\boldsymbol{e}_i, \quad {}_\alpha\boldsymbol{v}^+ = \frac{\partial_\alpha\boldsymbol{\chi}^+}{\partial_\alpha t^+}(\boldsymbol{X}, {}_\alpha t^+). \quad (2.23) $$

We note that in view of $(2.7)_1$, (2.16), $(2.23)_1$, $(2.18)_2$, $(2.15)_4$ and (2.22)

$$ \begin{aligned} {}_\alpha\mathrm{div}^+\,{}_\alpha\boldsymbol{T}^+ &= {}_\alpha\boldsymbol{Q}(t)\,{}_\alpha\mathrm{div}\,{}_\alpha\boldsymbol{T}, \\ {}_\alpha\dot{\boldsymbol{v}}^+ - {}_\alpha\boldsymbol{b}^+ &= {}_\alpha\boldsymbol{Q}(t)({}_\alpha\dot{\boldsymbol{v}} - {}_\alpha\boldsymbol{b}). \end{aligned} \quad (2.24) $$

It is convenient for the treatment of elastic–plastic bodies† to introduce nonsymmetric and symmetric Piola–Kirchhoff stress tensors denoted by ${}_\alpha\boldsymbol{P}$ and ${}_\alpha\boldsymbol{S}$, respectively, and related to the Cauchy stress tensor ${}_\alpha\boldsymbol{T}$ by

$$ {}_\alpha J{}_\alpha\boldsymbol{T} = {}_\alpha\boldsymbol{P}{}_\alpha\boldsymbol{F}^T = {}_\alpha\boldsymbol{F}{}_\alpha\boldsymbol{S}{}_\alpha\boldsymbol{F}^T. \quad (2.25) $$

†In [1, 2] both material and spatial descriptions of the basic field equations for elastic–plastic bodies can be found. Although in later sections of the present paper we shall mainly be using a material description, we have included the foregoing spatial description of the conservation laws and related equations in order to facilitate a comparison of our results, upon reduction to the elastic case, with those in [7].

The stress vector ${}_{\alpha}\boldsymbol{p}$ given by

$$ {}_{\alpha}\boldsymbol{p} = {}_{\alpha}\boldsymbol{P}_0\boldsymbol{n} \tag{2.26} $$

represents the surface force in the motion ${}_{\alpha}\chi$ but measured per unit area of the surface $\partial_0\mathscr{P}$ in the reference configuration ${}_0\boldsymbol{\kappa}$. Denoting by ${}_{\alpha}\boldsymbol{P}^+$ and ${}_{\alpha}\boldsymbol{S}^+$ the values taken on by ${}_{\alpha}\boldsymbol{P}$ and ${}_{\alpha}\boldsymbol{S}$ in the motion ${}_{\alpha}\chi^+$, it follows from (2.21), $(2.9)_2$ and (2.25) that

$$ {}_{\alpha}\boldsymbol{P}^+ = {}_{\alpha}\boldsymbol{Q}(t){}_{\alpha}\boldsymbol{P}, \quad {}_{\alpha}\boldsymbol{S}^+ = {}_{\alpha}\boldsymbol{S} \tag{2.27} $$

and hence

$$ {}_{\alpha}\dot{\boldsymbol{S}}^+ = {}_{\alpha}\dot{\boldsymbol{S}}. \tag{2.28} $$

Thus, $\boldsymbol{S}$ has the important property that neither itself nor its material time derivative is altered under superposed rigid body motions represented by the transformations (2.7).

The balance of linear momentum in the motion ${}_{\alpha}\chi$ may be written in terms of ${}_{\alpha}\boldsymbol{P}$ in the form

$$ \operatorname{Div} {}_{\alpha}\boldsymbol{P} + {}_0\rho\, {}_{\alpha}\boldsymbol{b} = {}_0\rho\, {}_{\alpha}\dot{\boldsymbol{v}}, \tag{2.29} $$

where Div is the (right) divergence operator with respect to X, having a component representation

$$ \operatorname{Div} {}_{\alpha}\boldsymbol{P} = \frac{\partial}{\partial X_B}({}_{\alpha}P_{iA}\boldsymbol{e}_i \otimes \boldsymbol{e}_A)\boldsymbol{e}_B = \frac{\partial_{\alpha}P_{iA}}{\partial X_A}\boldsymbol{e}_i. \tag{2.30} $$

It follows at once from $(2.15)_{1,4}$ and (2.29) that

$$ {}_{\alpha}J\, {}_{\alpha}\operatorname{div} {}_{\alpha}\boldsymbol{T} = \operatorname{Div} {}_{\alpha}\boldsymbol{P}. \tag{2.31} $$

Furthermore, in view of (2.27)

$$ \operatorname{Div} {}_{\alpha}\boldsymbol{P}^+ = {}_{\alpha}\boldsymbol{Q}(t) \operatorname{Div} {}_{\alpha}\boldsymbol{P}. \tag{2.32} $$

3. THE ELASTIC–PLASTIC SOLID

In this section we summarize the main ingredients of a purely mechanical rate-type theory of finitely deforming elastic–plastic solids. Our treatment is based on the alternative strain space formulation[12, 16] of the theory of plasticity originally proposed by Green and Naghdi [1, 2].

In addition to the kinematical and kinetical quantities introduced in Section 2, we assume the existence of a symmetric† second order tensor-valued function ${}^p_{\alpha}\boldsymbol{E} = {}^p_{\alpha}\boldsymbol{E}(X, t)$, called the *plastic strain* at (X, t) in the motion ${}_{\alpha}\chi$, and a scalar-valued function ${}_{\alpha}\kappa = {}_{\alpha}\kappa(X, t)$ called a *measure of work-hardening*.

For each α and at each X and t the values of strain ${}_{\alpha}\boldsymbol{E}(X, t)$, plastic strain ${}^p_{\alpha}\boldsymbol{E}(X, t)$ and work-hardening ${}_{\alpha}\kappa(X, t)$ in the motion ${}_{\alpha}\chi$ may be regarded as a point, which we denote by $\mathscr{U} = (\boldsymbol{E}, {}^p\boldsymbol{E}, \kappa)$, in a thirteen-dimensional Euclidean space $\mathscr{X}$ formed from the Cartesian product Sym × Sym × Real Line where Sym stands for the space of symmetric second order tensors.

We assume that both ${}^p_{\alpha}\boldsymbol{E}$ and ${}_{\alpha}\kappa$ are unaltered under superposed rigid body motions (2.7). Thus, let ${}^p_{\alpha}\boldsymbol{E}^+$ and ${}_{\alpha}\kappa^+$ be the plastic strain and measure of work-hardening associated with the motion ${}_{\alpha}\chi^+$. Employing the notation ${}_{\alpha}\mathscr{U} = ({}_{\alpha}\boldsymbol{E}, {}^p_{\alpha}\boldsymbol{E}, {}_{\alpha}\kappa)$,

†The symmetry of ${}^p_{\alpha}\boldsymbol{E}$ is shown in [16] to follow from a work assumption which we will be adopting below.

${}_{\alpha}\mathscr{U}^{+} = ({}_{\alpha}\boldsymbol{E}^{+}, {}_{\alpha}^{p}\boldsymbol{E}^{+}, {}_{\alpha}\kappa^{+})$ and recalling (2.10) and $(2.7)_2$ we may then write

$$ {}_{\alpha}\mathscr{U}^{+} = {}_{\alpha}\mathscr{U}, \quad {}_{\alpha}\dot{\mathscr{U}}^{+} = \frac{\partial({}_{\alpha}\mathscr{U}^{+})}{\partial_{\alpha}t^{+}} = {}_{\alpha}\dot{\mathscr{U}}. \tag{3.1} $$

We suppose that the stress ${}_{\alpha}\boldsymbol{S}$ in the motion ${}_{\alpha}\boldsymbol{\chi}$ is given by the constitutive equation

$$ {}_{\alpha}\boldsymbol{S} = \bar{\boldsymbol{S}}({}_{\alpha}\mathscr{U}). \tag{3.2} $$

We further assume that $\bar{\boldsymbol{S}}$ satisfies the symmetry condition†

$$ \left(\frac{\partial \bar{S}_{KL}}{\partial E_{MN}} - \frac{\partial \bar{S}_{MN}}{\partial E_{KL}}\right)\boldsymbol{e}_K \otimes \boldsymbol{e}_L \otimes \boldsymbol{e}_M \otimes \boldsymbol{e}_N = \boldsymbol{0}. \tag{3.3} $$

In view of (3.1) and (3.4), the stress ${}_{\alpha}\boldsymbol{S}^{+}$ in the motion ${}_{\alpha}\boldsymbol{\chi}^{+}$ is equal to ${}_{\alpha}\boldsymbol{S}$ and the invariance condition $(2.27)_2$ is satisfied.

We also assume the existence of a continuously differentiable scalar-valued *yield* or *loading* function g on $\mathscr{X}$. Fixed values of ${}^{p}\boldsymbol{E}$ and κ give a six-dimensional subspace of $\mathscr{X}$, called *the strain space at* $({}^{p}\boldsymbol{E}, \kappa)$. We assume that for fixed values of ${}^{p}\boldsymbol{E}$ and κ the equation

$$ g(\mathscr{U}) = 0 \tag{3.4} $$

represents a smooth orientable closed hypersurface $\partial\mathscr{E}$ of dimension five enclosing a region $\mathscr{E}$ of the strain space at $({}^{p}\boldsymbol{E}, \kappa)$. The work-hardening parameter is chosen so that $g(\mathscr{U}) < 0$ for all points in the interior of $\mathscr{E}$. The hypersurface $\partial\mathscr{E}$ is called *the yield surface at* $({}^{p}\boldsymbol{E}, \kappa)$, while the interior of $\mathscr{E}$ is called *the elastic region at* $({}^{p}\boldsymbol{E}, \kappa)$. As different values of $({}^{p}\boldsymbol{E}, \kappa)$ are taken on by the particle X as a result of the motion of the elastic–plastic body, the surface $\partial\mathscr{E}$ sweeps out a hypersurface of dimension twelve in $\mathscr{X}$, all points of which satisfy $g = 0$. We denote this hypersurface by Σ. See Fig. 1.

If corresponding to a motion ${}_{\alpha}\boldsymbol{\chi}$, a point ${}_{\alpha}\mathscr{U}$ satisfies (3.4), then since ${}_{\alpha}\mathscr{U}$ is unaltered under the transformations (2.7), equation (3.4) will still be satisfied in the motion ${}_{\alpha}\boldsymbol{\chi}^{+}$. Consequently, the notion of a yield surface and also that of an elastic region are invariant notions.

Corresponding to a motion ${}_{\alpha}\boldsymbol{\chi}$, there will be associated with each particle X of the elastic–plastic body a continuous oriented curve, or trajectory, ${}_{\alpha}C$ in $\mathscr{X}$ consisting of

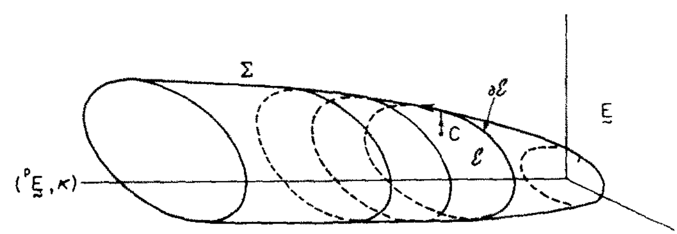

Fig. 1. A schematic diagram illustrating how $\partial\mathscr{E}$, the yield surface in strain space at $({}^{p}\boldsymbol{E}, \kappa)$, sweeps out the hypersurface Σ in the 13-dimensional space of $\mathscr{X} = (\boldsymbol{E}, {}^{p}\boldsymbol{E}, \kappa)$. All planes perpendicular to the $({}^{p}\boldsymbol{E}, \kappa)$ axis represent strain space and the strain space at zero value of $({}^{p}\boldsymbol{E}, \kappa)$ is explicitly denoted by $\boldsymbol{E}$. Also shown is a typical trajectory C in $\mathscr{X}$.

†This is equivalent to the condition that $\boldsymbol{S}$ be derivable from a potential, as indeed is the case in the general thermodynamical theory (see Section 4 of [2]). The purely mechanical theory may be regarded as corresponding to the isothermal case of the general theory. (The existence of a potential can also be demonstrated by an argument based on the work postulate put forward by Naghdi and Trapp[16] and introduced later in this section.)

points ${}_\alpha\mathscr{U}(X,t) = ({}_\alpha E(X,t), {}^p_\alpha E(X,t), {}_\alpha\kappa(X,t))$ and parameterized by time t. These trajectories are restricted to lie initially in the elastic region or on its boundary $\partial\mathscr{E}$ in the strain space at the initial value of $({}^p_\alpha E, {}_\alpha\kappa)$, i.e.

$$g({}_\alpha\mathscr{U}) \leqslant 0 \tag{3.5}$$

initially on ${}_\alpha C$.

Constitutive equations for ${}^p_\alpha\dot{E}$ and ${}_\alpha\dot{\kappa}$ may be written by introducing an undetermined scalar-valued function $\lambda(\mathscr{U})$ on $\mathscr{X}$ together with symmetric second order tensor-valued constitutive functions $\rho(\mathscr{U})$ and $\mathscr{C}(\mathscr{U})$ such that

$${}_\alpha\dot{\kappa} = \mathscr{C}({}_\alpha\mathscr{U})\cdot{}^p_\alpha\dot{E} = \mathscr{C}_{MN}\,{}^p_\alpha\dot{E}_{MN} \tag{3.6}$$

and

$${}^p_\alpha\dot{E} = \begin{cases} \mathbf{0} & \text{if } g({}_\alpha\mathscr{U}) < 0 \quad \text{(a)} \\ \mathbf{0} & \text{if } g({}_\alpha\mathscr{U}) = 0 \text{ and } {}_\alpha\hat{g} < 0 \quad \text{(b)} \\ \mathbf{0} & \text{if } g({}_\alpha\mathscr{U}) = 0 \text{ and } {}_\alpha\hat{g} = 0 \quad \text{(c)} \\ \lambda({}_\alpha\mathscr{U})\,{}_\alpha\hat{g}\,\rho({}_\alpha\mathscr{U}) & \text{if } g({}_\alpha\mathscr{U}) = 0 \text{ and } {}_\alpha\hat{g} > 0 \quad \text{(d)} \end{cases} \tag{3.7}$$

where

$${}_\alpha\hat{g} = \frac{\partial g}{\partial E}({}_\alpha\mathscr{U})\cdot{}_\alpha\dot{E} = \frac{\partial g}{\partial E_{MN}}\,{}_\alpha\dot{E}_{MN}, \tag{3.8}$$

and where the notation $\partial g/\partial E_{MN}$ stands for the symmetric form $\frac{1}{2}(\partial g/\partial E_{MN} + \partial g/\partial E_{NM})$. We observe that ${}_\alpha\dot{E} = \mathbf{0}$ for $\alpha = 0$ implies that ${}_\alpha\hat{g} = 0$ for $\alpha = 0$. Thus ${}^p_0\dot{E} = \mathbf{0}$, that is, the plastic strain must always be time-independent in the identity motion; it retains its initial value at each particle of the body. Furthermore, in view of (3.6), ${}_\alpha\dot{\kappa} = 0$ for $\alpha = 0$.

The conditions involving g and $\hat{g}$ in (3.7) are the loading criteria of the strain space formulation of plasticity. The four cases (a)–(d) in (3.7) are said to represent *an elastic state, unloading from an elastic–plastic state, neutral loading from an elastic–plastic state* and *loading from an elastic–plastic state*, respectively.

Equations (3.6) and (3.7) are called constitutive equations of the *rate-type*, since they involve time rates of ${}_\alpha E$, ${}^p_\alpha E$ and ${}_\alpha\kappa$. However, both ${}^p_\alpha\dot{E}$ and ${}_\alpha\dot{\kappa}$ as given by (3.7) and (3.6) are independent of the time scale used to compute these rates since the equations are linear and hence homogeneous of degree one in the rates. Furthermore, neither the yield function g, nor the function $\tilde{S}$ in (3.2), depend on rates. For these reasons the present theory is called *rate-independent*. It is intended to apply to inviscid elastic–plastic materials in which time effects such as creep and relaxation may be ignored.

The conditions involving g and $\hat{g}$ on the r.h.s. of (3.7) are invariant statements. This may be seen at once from $(3.1)_1$, $(2.14)_2$ and the observation that

$${}_\alpha\hat{g}^+ = \frac{\partial g}{\partial E^+}({}_\alpha\mathscr{U}^+)\cdot{}_\alpha\dot{E}^+ = {}_\alpha\hat{g}. \tag{3.9}$$

Thus, not only will the same trajectory in $\mathscr{X}$ and the same yield surface be associated with the motions ${}_\alpha\chi^+$ and ${}_\alpha\chi$, but in addition an elastic state, unloading, neutral loading and loading, respectively, will occur at $(X, {}_\alpha t^+)$ if and only if the same state occurs at (X, t). We also note that in view of $(3.1)_1$ and (3.9) the constitutive equations (3.6) and (3.7) are properly invariant statements, satisfying the invariance requirement $(3.1)_2$.

We assume that the coefficient of ${}_\alpha\hat{g}$ in (3.7d) is nonzero on the yield surface. Then, without loss of generality we may write†

$$\boldsymbol{\rho} \neq \mathbf{0}, \quad \lambda > 0. \tag{3.10}$$

Next, we stipulate that loading from an elastic–plastic state must lead to an elastic–plastic state‡. Therefore in any motion ${}_{\alpha}\boldsymbol{\chi}$,

$$ {}_{\alpha}\dot{g} = \frac{\partial g}{\partial \boldsymbol{E}}({}_{\alpha}\mathscr{U})\cdot {}_{\alpha}\dot{\boldsymbol{E}} + \frac{\partial g}{\partial {}^{p}\boldsymbol{E}}({}_{\alpha}\mathscr{U})\cdot {}^{p}_{\alpha}\dot{\boldsymbol{E}} + \frac{\partial g}{\partial \kappa}({}_{\alpha}\mathscr{U}){}_{\alpha}\dot{\kappa} = 0 \tag{3.11}$$

whenever loading occurs. With the use of (3.6), (3.7d), (3.8) and (3.10), we deduce from (3.11) that

$$1 + \lambda(\mathscr{U})\boldsymbol{\rho}(\mathscr{U})\cdot\left\{\frac{\partial g}{\partial {}^{p}\boldsymbol{E}}(\mathscr{U}) + \frac{\partial g}{\partial \kappa}(\mathscr{U})\mathscr{C}(\mathscr{U})\right\} = 0 \tag{3.12}$$

at all points of $\partial\mathscr{E}$ through which loading can occur. We regard (3.12) as an equation for the determination of λ.

We now briefly discuss a geometrical interpretation of the conditions (3.7). For an illustration see Fig. 1. It follows from (3.4), (3.6) and (3.7a) that in an elastic state the trajectory ${}_{\alpha}$C in $\mathscr{X}$ lies in the interior of the region $\mathscr{E}$ of the strain space at $({}^{p}_{\alpha}\boldsymbol{E}, \kappa)$, and that ${}_{\alpha}$C must remain in the strain space at $({}^{p}_{\alpha}\boldsymbol{E}, \kappa)$ until the yield surface $\partial\mathscr{E}$ at $({}^{p}_{\alpha}\boldsymbol{E}, {}_{\alpha}\kappa)$ is reached. Similarly, by (3.4), (3.6), (3.7b) and (3.8), during unloading from an elastic–plastic state the trajectory ${}_{\alpha}$C intersects $\partial\mathscr{E}$, the components of its tangent vector ${}_{\alpha}\dot{\mathscr{U}} = ({}_{\alpha}\dot{\boldsymbol{E}}, {}^{p}_{\alpha}\dot{\boldsymbol{E}}, {}_{\alpha}\dot{\kappa})$ perpendicular to the strain space at $({}^{p}_{\alpha}\boldsymbol{E}, {}_{\alpha}\kappa)$ being zero, while the inner product of ${}_{\alpha}\dot{\boldsymbol{E}}$ and the outward unit normal $\partial g/\partial \boldsymbol{E}({}_{\alpha}\mathscr{U})$ (at $g = 0$) to $\partial\mathscr{E}$ is negative. Therefore, ${}_{\alpha}$C remains in the strain space at $({}^{p}_{\alpha}E, {}_{\alpha}\kappa)$ and is moving into the interior of $\mathscr{E}$. From the expression for ${}_{\alpha}\dot{g}$ in $(3.11)_1$ it is clear that ${}_{\alpha}g$ is decreasing in value during unloading. As regards neutral loading from an elastic–plastic state, it follows from (3.4), (3.6), (3.7c), (3.8) and (3.11) that ${}_{\alpha}$C intersects $\partial\mathscr{E}$ at $({}^{p}_{\alpha}\boldsymbol{E}, {}_{\alpha}\kappa)$; the components of its tangent perpendicular to the strain space at $({}^{p}_{\alpha}\boldsymbol{E}, {}_{\alpha}\kappa)$ are zero; its component ${}_{\alpha}\dot{\boldsymbol{E}}$ is perpendicular to the normal $\partial g/\partial \boldsymbol{E}$; and ${}_{\alpha}g$ is stationary: Therefore, ${}_{\alpha}$C continutes to move in the yield surface $\partial\mathscr{E}$ at $({}^{p}_{\alpha}\boldsymbol{E}, {}_{\alpha}\kappa)$. It is only during loading from an elastic–plastic state that the trajectory ${}_{\alpha}$C can leave the strain space at $({}^{p}_{\alpha}\boldsymbol{E}, {}_{\alpha}\kappa)$. Indeed it follows from (3.7d), (3.8) and (3.11) that during loading ${}_{\alpha}$C intersects $\partial\mathscr{E}$ and locally moves outwardly with it in the sense that the inner product of ${}_{\alpha}\dot{\boldsymbol{E}}$ and $\partial g/\partial \boldsymbol{E}$ is positive. The components $({}^{p}_{\alpha}\dot{\boldsymbol{E}}, {}_{\alpha}\dot{\kappa})$ of the tangent perpendicular to the strain space at $({}^{p}_{\alpha}\boldsymbol{E}, {}_{\alpha}\kappa)$ are given by (3.6) and (3.7d). During loading the trajectory ${}_{\alpha}$C remains in the hypersurface Σ swept out by $\partial\mathscr{E}$ as ${}_{\alpha}\boldsymbol{E}$, ${}^{p}_{\alpha}\boldsymbol{E}$ and ${}_{\alpha}\kappa$ change with time. Finally, recalling the consistency condition as well as (3.7a–c), it is clear that the restriction (3.5) holds for all time and that the trajectory ${}_{\alpha}$C remains within the region enclosed by Σ or lies on Σ.

The mechanical theory of elastic–plastic materials under consideration involves a set of five functions consisting of the motion ${}_{\alpha}\boldsymbol{\chi}$, the stress tensor ${}_{\alpha}\boldsymbol{S}$, the variables ${}^{p}_{\alpha}\boldsymbol{E}$ and ${}_{\alpha}\kappa$, and the body force ${}_{\alpha}\boldsymbol{b}$. If ${}^{p}_{\alpha}\boldsymbol{E}$ and ${}_{\alpha}\kappa$ are given initial values at time t°, the motion ${}_{\alpha}\boldsymbol{\chi}$ can be used to determine ${}_{\alpha}\boldsymbol{E}$, and assuming sufficient smoothness, the differential equations (3.6) and (3.7) can be solved for ${}^{p}_{\alpha}\boldsymbol{E}$ and ${}_{\alpha}\kappa$ in an interval containing t°. The new position of $\partial\mathscr{E}$ can be found from (3.4), the stress tensor ${}_{\alpha}\boldsymbol{S}$ from (3.2) and the Cauchy stress from (2.25). The field equation $(2.15)_4$ is assumed to hold for any choice of ${}_{\alpha}\boldsymbol{\chi}$ and may be used to calculate ${}_{\alpha}\boldsymbol{b}$, while ${}_{\alpha}\rho$ may be found using (2.15) together with a prescribed reference mass density.

†We are assuming that for given values of ${}^{p}\boldsymbol{E}$ and κ and a given continuous function $\boldsymbol{\rho}$, a loading function may be chosen independently of $\boldsymbol{\rho}$ such that the associated yield surface $\partial\mathscr{E}$ is the boundary of an arbitrarily small neighborhood of the strain space at $({}^{p}\boldsymbol{E}, \kappa)$. If there is a region of the strain-space at $({}^{p}\boldsymbol{E}, \kappa)$ which $\partial\mathscr{E}$ cannot traverse, then $\boldsymbol{\rho}$ and λ are unrestricted in this region. For particular materials this may be acceptable. $\boldsymbol{\rho}$ and λ must, however, satisfy (3.10) at all points that $\partial\mathscr{E}$ may traverse. For a given loading function g, the hypersurface $g = 0$ will be the fixed twelve-dimensional hypersurface Σ in the space $\mathscr{X}$. It is actually sufficient that λ and $\boldsymbol{\rho}$ be defined only on Σ.

‡This is the "consistency" condition in the context of the present strain space formulation of plasticity theory.

We now turn to the work assumption introduced by Naghdi and Trapp in [16] and further examined by them in [17]. We recall that a motion ${}_\alpha\chi$ is said to be a homogeneous motion if and only if its deformation gradient, given by $(2.2)_1$, is independent of X, i.e. $(\partial_\alpha F/\partial X) = 0$. The strain tensor ${}_\alpha E$ in $(2.4)_2$ will then also be independent of X. If the body $\mathscr{B}$ is homogeneous in its reference configuration ${}_0\kappa$, that is, if the mass density ${}_0\rho$ and all of the constitutive functions do not depend explicitly on X, then, in any homogeneous motion of $\mathscr{B}$, the stress ${}_\alpha S$, the plastic strain ${}^p_\alpha E$ and work-hardening parameter ${}_\alpha\kappa$ will also be independent of X. We say that a motion ${}_\alpha\chi$ is a homogeneous cycle in a closed time interval $[t_1, t_2]$, $t_1 < t_2$, if it is homogeneous and if for each particle of the body $\mathscr{B}$ the position ${}_\alpha x$ and the strain tensor ${}_\alpha E$ assume the same values at t_1 and t_2. We designate such a smooth homogeneous cycle by ${}_\alpha c[t_1, t_2]$. The work assumption of Naghdi and Trapp then is: The work done on the elastic–plastic body $\mathscr{B}$ by surface tractions and body forces in any smooth homogeneous cycle ${}_\alpha c[t_1, t_2]$ is nonnegative. We emphasize that such a cycle always exists, being maintained by a suitable choice of the body force field†. It follows from the work assumption of Naghdi and Trapp that‡[16].

$$\left(\frac{\partial \bar{S}}{\partial {}^pE}(\mathscr{U}) + \frac{\partial \bar{S}}{\partial \kappa}(\mathscr{U})\otimes\mathscr{C}(\mathscr{U})\right)[\rho(\mathscr{U})] = -\gamma(\mathscr{U})\frac{\partial g}{\partial E}(\mathscr{U}) \tag{3.13a}$$

or, in indicial notation,

$$\left(\frac{\partial \bar{S}_{KL}}{\partial {}^pE_{MN}}(\mathscr{U}) + \frac{\partial \bar{S}_{KL}}{\partial \kappa}(\mathscr{U})\mathscr{C}_{MN}(\mathscr{U})\right)\rho_{MN}(\mathscr{U}) = -\gamma(\mathscr{U})\frac{\partial g}{\partial E_{KL}}(\mathscr{U}) \tag{3.13b}$$

evaluated on $\partial\mathscr{E}$, the yield surface at $({}^pE, \kappa)$, where the scalar function γ satisfies§

$$\gamma(\mathscr{U}) \geqslant 0 \tag{3.14}$$

on $\mathscr{Z}$. We emphasize that (3.13) holds even for a motion that is not homogeneous.¶ We note that (3.13) is an invariant statement.

4. THE MOTION ${}_\alpha\chi^*$ AND THE INVARIANCE PROPERTIES ASSOCIATED WITH IT

In this section we employ a method introduced recently by Casey and Naghdi [7] as a means of constructing properly invariant infinitesimal theories. We recall that from among the particles of $\mathscr{B}$, one, denoted by Y and called a *pivot* is chosen. Then, by (2.1)–(2.3), we have

$${}_\alpha y = {}_\alpha\chi(Y, t), \quad \frac{\partial_\alpha \chi}{\partial X}(Y, t) = {}_\alpha R(Y, t)_\alpha U(Y, t) \tag{4.1}$$

where $Y = {}_0y = {}_0\kappa(Y)$. For any motion ${}_\alpha\chi$ we can construct a motion ${}_\alpha\chi^* = \pi({}_\alpha\chi)$ by removing from ${}_\alpha\chi$ the rotation and translation at the pivot Y, while maintaining at all particles of $\mathscr{B}$ the stretch (and hence finite strain) experienced in the motion ${}_\alpha\chi$. The motion ${}_\alpha\chi^*$ is given by [7]

$${}_\alpha x^* = {}_\alpha\chi^*(X, t^*) = {}_\alpha R^T(Y, t)\{{}_\alpha\chi(X, t) - {}_\alpha\chi(Y, t)\} + Y, \quad t^* = t - c \tag{4.2}$$

where‖ c is a real constant. The configuration of $\mathscr{B}$ at time t^* in the motion ${}_\alpha\chi^*$ is the mapping ${}_\alpha\kappa^*$ given by ${}_\alpha\kappa^* = {}_\alpha\chi^* o_0\kappa$. The image of ${}_0\mathscr{R}$ in the motion ${}_\alpha\chi^*$ is denoted by ${}_\alpha\mathscr{R}^* = {}_\alpha\chi^*({}_0\mathscr{R}, t^*)$ and its boundary is denoted by $\partial_\alpha\mathscr{R}^*$. Clearly, in the case of the identity

†In general, a cycle ${}_\alpha c[t_1, t_2]$ will not correspond to a *cyclic elastic–plastic* process, that is, a process in which ${}_\alpha\mathscr{U}(t)$ takes on the same values at times t_1 and t_2.

‡Equation (3.13) is equivalent to (5.4) of [16]. See [14] for a discussion of this point.

§See the first footnote on p. 1124.

¶For a discussion, see ([16], p. 40) or ([17], p. 63).

‖With a slight increase in generality, we could, as in [7], use a different t^* for each ${}_\alpha\chi^*$.

motion, (4.2) gives ${}_0X^* = X$, so that ${}_0\chi^* = {}_0\chi$ and ${}_0\kappa^* = {}_0\kappa$. We note that (4.2) is of the form (2.7) and that ${}_\alpha\chi^*$ is therefore a member of the class containing all the motions ${}_\alpha\chi^+$.

Next we define "difference" motions χ' and $\chi^{*\prime}$ by the relations

$$\chi' = {}_2\chi \circ {}_1\chi^{-1}, \qquad \chi^{*\prime} = {}_2\chi^* \circ ({}_1\chi^*)^{-1} \tag{4.3}$$

so that

$${}_2x = \chi'({}_1x, t), \quad {}_2x^* = \chi^{*\prime}({}_1x^*, t^*). \tag{4.4}$$

We say that the "superposition" of the motion χ' on ${}_1\chi$ produces the motion ${}_2\chi$. Similarly, ${}_2\chi^*$ differs from ${}_1\chi^*$ by a "superposed" or "difference" motion $\chi^{*\prime}$.

Relative to the reference configuration ${}_0\kappa$, the deformation gradient, displacement and displacement gradient of the motions ${}_\alpha\chi^*$ are given, as in (2.2), (2.5) and (2.6), by

$${}_\alpha F^* = \frac{\partial_\alpha \chi^*}{\partial X}(X, t), \quad {}_\alpha u^* = ({}_\alpha\chi^* - {}_0\chi)(X, t^*) = {}_\alpha x^* - X, \quad {}_\alpha G^* = {}_\alpha F^* - I. \tag{4.5}$$

Of course, ${}_0F^* = I$, ${}_0u^* = o$ and ${}_0G^* = 0$. Likewise, as in (2.4), we define Cauchy–Green stretch tensors ${}_\alpha C^*$ and strain tensors ${}_\alpha E^*$ by

$${}_\alpha C^* = ({}_\alpha F^*)^T {}_\alpha F^*, \quad {}_\alpha E^* = \tfrac{1}{2}({}_\alpha C^* - I). \tag{4.6}$$

For $\alpha = 0$, ${}_0C^* = I$, ${}_0E^* = 0$.

It follows from $(4.5)_1$, (4.2), (2.3), $(2.2)_2$ and the proper orthogonality of ${}_\alpha R$ that

$${}_\alpha F^* = {}_\alpha R^T(Y, t)_\alpha F, \quad {}_\alpha J^* = \det({}_\alpha F^*) = {}_\alpha J > 0. \tag{4.7}$$

In view of $(4.7)_2$, at each t^* the mappings ${}_\alpha F^*$ and ${}_\alpha\chi^*$ are all invertible.

An application of the chain rule of differentiation to $(4.3)_1$ yields

$$F' = F'({}_1x, t) = \frac{\partial \chi'}{\partial_1 x}({}_1x, t) = {}_2F_1F^{-1}, \tag{4.8}$$

while, similarly, it follows from $(4.3)_2$ that

$$F^{*\prime} = \frac{\partial \chi^{*\prime}}{\partial_1 x^*}({}_1x^*, t^*) = {}_2F^*({}_1F^*)^{-1}. \tag{4.9}$$

Clearly, F' and $F^{*\prime}$ are both invertible. It follows from $(4.7)_1$, (4.8) and (4.9) that

$$F^{*\prime}({}_1x^*, t) = {}_2R^T(Y, t)F'({}_1x, t)_1R(Y, t). \tag{4.10}$$

In view of the invertibility of ${}_\alpha F^*$, we may apply the polar decomposition theorem to obtain

$${}_\alpha F^* = {}_\alpha R^* {}_\alpha U^*, \tag{4.11}$$

with ${}_\alpha R^*$ proper orthogonal and ${}_\alpha U^*$ symmetric positive definite. It follows from (4.6), (4.7), (4.11), (2.3), (2.4) and $(4.2)_2$ that

$$\begin{gathered} {}_\alpha C^* = ({}_\alpha U^*)^2 = {}_\alpha C, \quad {}_\alpha U^* = {}_\alpha U, \quad {}_\alpha E^* = {}_\alpha E, \\ {}_\alpha R^* = {}_\alpha R^T(Y, t)_\alpha R, \\ {}_\alpha \dot{C}^* = \frac{\partial_\alpha C^*}{\partial t^*} = \frac{\partial_\alpha C}{\partial t} = {}_\alpha \dot{C}, \quad {}_\alpha \dot{E}^* = {}_\alpha \dot{E}. \end{gathered} \tag{4.12}$$

When $X = Y$, from (4.2) we obtain

$$ {}_{\alpha}y^{*} = {}_{\alpha}\chi^{*}(Y, t^{*}) = Y, \tag{4.13}$$

so that the particle Y retains its reference position Y in the configuration ${}_{\alpha}\kappa^{*}$. Then, $(4.2)_1$ may be rewritten as

$$ {}_{\alpha}\chi(X, t) - {}_{\alpha}\chi(Y, t) = {}_{\alpha}R(Y, t)({}_{\alpha}x^{*} - {}_{\alpha}y^{*}). \tag{4.14}$$

Thus, the configuration ${}_{\alpha}\kappa$ is related to ${}_{\alpha}\kappa^{*}$ by a superposed rigid motion whose deformation gradient relative to ${}_{\alpha}\kappa^{*}$ is ${}_{\alpha}R(Y, t)$.

Again, setting $X = Y$ in (4.11) and $(4.12)_{2,4}$ leads to the equations

$$ {}_{\alpha}R^{*}(Y, t^{*}) = I, \quad {}_{\alpha}F^{*}(Y, t^{*}) = {}_{\alpha}U(Y, t), \tag{4.15}$$

so that the particle Y always experiences pure stretch in the motion ${}_{\alpha}\chi^{*}$.

Let ${}^{p}_{\alpha}E^{*}$ and ${}_{\alpha}\kappa^{*}$ be the plastic strain and measure of work-hardening associated with the motion ${}_{\alpha}\chi^{*}$. We employ the notation ${}_{\alpha}\mathscr{U}^{*} = ({}_{\alpha}E^{*}, {}^{p}_{\alpha}E^{*}, {}_{\alpha}\kappa^{*})$. Since the motions ${}_{\alpha}\chi^{*}$ and ${}_{\alpha}\chi$ are related through a superposed rigid body motion it follows from (3.1) that

$$ {}_{\alpha}\mathscr{U}^{*} = {}_{\alpha}\mathscr{U} \tag{4.16}$$

where $(4.12)_3$ has also been recalled. Consequently

$$ {}_{\alpha}\dot{\mathscr{U}}^{*} = \frac{\partial}{\partial t^{*}}({}_{\alpha}\mathscr{U}^{*}) = {}_{\alpha}\dot{\mathscr{U}}. \tag{4.17}$$

In view of $(4.12)_3$ and (4.16), it is clear that the same trajectory in $\mathscr{X}$, consisting of points

$$\begin{aligned} {}_{\alpha}\mathscr{U}^{*}(X, t^{*}) &= ({}_{\alpha}E^{*}(X, t^{*}), {}^{p}_{\alpha}E^{*}(X, t^{*}), {}_{\alpha}\kappa^{*}(X, t^{*})) \\ &= ({}_{\alpha}E(X, t), {}^{p}_{\alpha}E(X, t), {}_{\alpha}\kappa(X, t)) = {}_{\alpha}\mathscr{U}(X, t) \end{aligned} \tag{4.18}$$

is associated with the motions ${}_{\alpha}\chi^{*}$ and ${}_{\alpha}\chi$. (This is a special instance of the invariance property of such a trajectory, alluded to following (3.9).) Furthermore, since the loading conditions are invariant statements, an elastic state, unloading, neutral loading and loading at (X, t^{*}) in the motion ${}_{\alpha}\chi^{*}$ correspond exactly to an elastic state, unloading, neutral loading and loading at (X, t) in the motion ${}_{\alpha}\chi$, and the same yield surface will be associated with ${}_{\alpha}\chi^{*}$ and ${}_{\alpha}\chi$ (i.e. $\partial_{\alpha}\Sigma^{*} = \partial_{\alpha}\Sigma$). For later convenience, we will state these conditions explicitly.

First, we note that it follows from $(4.12)_{3,6}$, (4.16) and (3.18) that

$$ {}_{\alpha}\hat{g}^{*} = \frac{\partial g}{\partial E^{*}}({}_{\alpha}\mathscr{U}^{*}) \cdot {}_{\alpha}\dot{E}^{*} = {}_{\alpha}\hat{g}. \tag{4.19}$$

This is, of course, a special case of the result (3.9). We may use $(4.12)_3$, (4.16), (4.17), (4.19), (3.7) and (3.6) to write the loading conditions in the following form:

$$ {}^{p}_{\alpha}\dot{E}^{*} = {}^{p}_{\alpha}\dot{E} = \begin{cases} 0 & \text{if } g({}_{\alpha}\mathscr{U}^{*}) < 0 & \text{(a)} \\ 0 & \text{if } g({}_{\alpha}\mathscr{U}^{*}) = 0 \text{ and } {}_{\alpha}\hat{g}^{*} < 0 & \text{(b)} \\ 0 & \text{if } g({}_{\alpha}\mathscr{U}^{*}) = 0 \text{ and } {}_{\alpha}\hat{g}^{*} = 0 & \text{(c)} \\ \lambda({}_{\alpha}\mathscr{U}^{*}){}_{\alpha}\hat{g}^{*}\rho({}_{\alpha}\mathscr{U}^{*}) & \text{if } g({}_{\alpha}\mathscr{U}^{*}) = 0 \text{ and } {}_{\alpha}\hat{g}^{*} > 0 & \text{(d)} \end{cases} \tag{4.20}$$

and

$$
{}_\alpha\dot{\kappa}^* = \begin{cases} 0 \text{ in cases (4.20a,b,c)} & \text{(a)} \\ \lambda({}_\alpha\mathscr{U}^*){}_\alpha\hat{g}^*\mathscr{C}({}_\alpha\mathscr{U}^*)\cdot\rho({}_\alpha\mathscr{U}^*) \text{ in case (4.20d).} & \text{(b)} \end{cases} \tag{4.21}
$$

We note that if the consistency condition is invoked for loading in the motion ${}_\alpha\chi^*$, again (3.12) results.

Let ${}_\alpha\boldsymbol{n}^*$ be the outward unit normal vector to the surface $\partial_\alpha\mathscr{P}^* = {}_\alpha\chi^*(\partial_0\mathscr{P}, t^*)$ and let ${}_\alpha\boldsymbol{t}^*$ and ${}_\alpha\boldsymbol{T}^*$ be the corresponding Cauchy stress vector and stress tensor, respectively. Then

$$
{}_\alpha\boldsymbol{t}^* = {}_\alpha\boldsymbol{T}^*{}_\alpha\boldsymbol{n}^*, \quad ({}_\alpha\boldsymbol{T}^*)^T = {}_\alpha\boldsymbol{T}^*. \tag{4.22}
$$

Since ${}_\alpha\chi^*$ is related to ${}_\alpha\chi$ by a superposed rigid motion whose deformation gradient relative to position in the configuration ${}_\alpha\boldsymbol{\kappa}$ is ${}_\alpha\boldsymbol{R}^T(Y, t)$, it follows from (2.19) that

$$
{}_\alpha\boldsymbol{n}^* = {}_\alpha\boldsymbol{R}^T(Y, t){}_\alpha\boldsymbol{n} \tag{4.23}
$$

while, in view of (2.20),

$$
{}_\alpha\boldsymbol{t}^* = {}_\alpha\boldsymbol{R}^T(Y, t){}_\alpha\boldsymbol{t} \tag{4.24}
$$

and, by (2.21)

$$
{}_\alpha\boldsymbol{T}^* = {}_\alpha\boldsymbol{R}^T(Y, t){}_\alpha\boldsymbol{T}{}_\alpha\boldsymbol{R}(Y, t). \tag{4.25}
$$

The nonsymmetric and symmetric Piola–Kirchhoff stress tensors associated with the motion ${}_\alpha\chi^*$ are denoted by ${}_\alpha\boldsymbol{P}^*$ and ${}_\alpha\boldsymbol{S}^*$, respectively. Then, as in (2.25),

$$
{}_\alpha J^*{}_\alpha\boldsymbol{T}^* = {}_\alpha\boldsymbol{P}^*({}_\alpha\boldsymbol{F}^*)^T = {}_\alpha\boldsymbol{F}^*{}_\alpha\boldsymbol{S}^*({}_\alpha\boldsymbol{F}^*)^T. \tag{4.26}
$$

Similarly, the stress vector ${}_\alpha\boldsymbol{p}^*$ is given by

$$
{}_\alpha\boldsymbol{p}^* = {}_\alpha\boldsymbol{P}^*{}_0\boldsymbol{n}. \tag{4.27}
$$

It follows from (4.7), (4.8), (4.25), (4.26) and (2.25) that

$$
{}_\alpha\boldsymbol{P}^* = {}_\alpha\boldsymbol{R}^T(Y, t){}_\alpha\boldsymbol{P}, \quad {}_\alpha\boldsymbol{S}^* = {}_\alpha\boldsymbol{S}. \tag{4.28}
$$

These equations are, of course, special cases of (2.27). It follows from $(4.28)_2$ and $(4.2)_2$ that

$$
{}_\alpha\dot{\boldsymbol{S}}^* = \frac{\partial_\alpha\boldsymbol{S}^*}{\partial t^*} = {}_\alpha\dot{\boldsymbol{S}}, \tag{4.29}
$$

a special case of (2.28).

From (4.27), (4.28) and (2.26) we obtain the equation

$$
{}_\alpha\boldsymbol{p}^* = {}_\alpha\boldsymbol{R}^T(Y, t){}_\alpha\boldsymbol{p}. \tag{4.30}
$$

For an elastic–plastic material, ${}_\alpha\boldsymbol{S}^*$ satisfies the relations

$$
{}_\alpha\boldsymbol{S}^* = \bar{\boldsymbol{S}}({}_\alpha\mathscr{U}^*) = \bar{\boldsymbol{S}}({}_\alpha\mathscr{U}) = {}_\alpha\boldsymbol{S} \tag{4.31}
$$

where (3.2) and (4.16) have been invoked. Denoting the mass density in the configuration ${}_\alpha\boldsymbol{\kappa}^*$ by ${}_\alpha\rho^*$ we obtain

$$
{}_\alpha\rho^*{}_\alpha J^* = {}_0\rho, \quad {}_\alpha\rho^* = {}_\alpha\rho, \tag{4.32}
$$

which are special cases of (2.18).

Let ${}_\alpha\boldsymbol{b}^*$ be the body force per unit mass in the configuration ${}_\alpha\boldsymbol{\kappa}^*$ and let

$$ {}_\alpha\boldsymbol{v}^* = \frac{\partial_\alpha \boldsymbol{\chi}^*}{\partial t^*}(X, t^*) \tag{4.33}$$

be the particle velocity in the motion ${}_\alpha\boldsymbol{\chi}^*$. The local form of the balance of linear momentum in the motion ${}_\alpha\boldsymbol{\chi}^*$ is

$$ {}_\alpha\mathrm{div}^*\, {}_\alpha\boldsymbol{T}^* + {}_\alpha\rho^*\, {}_\alpha\boldsymbol{b}^* = {}_\alpha\rho^*\, {}_\alpha\dot{\boldsymbol{v}}^* \tag{4.34}$$

where ${}_\alpha\mathrm{div}^*$ is the divergence operator with respect to position in the configuration ${}_\alpha\boldsymbol{\kappa}^*$ being defined in a manner paralleling $(2.23)_1$. It is readily seen that

$$\begin{aligned} {}_\alpha\mathrm{div}^*\, {}_\alpha\boldsymbol{T}^* &= {}_\alpha\boldsymbol{R}^T(Y, t)_\alpha\, \mathrm{div}\, {}_\alpha\boldsymbol{T}, \\ {}_\alpha\dot{\boldsymbol{v}}^* - {}_\alpha\boldsymbol{b}^* &= {}_\alpha\boldsymbol{R}^T(Y, t)({}_\alpha\dot{\boldsymbol{v}} - {}_\alpha\boldsymbol{b}) \end{aligned} \tag{4.35}$$

which are special cases of (2.24).

As in (2.29) the balance of linear momentum in the motion ${}_\alpha\boldsymbol{\chi}^*$ may be written in terms of ${}_\alpha\boldsymbol{P}^*$ with

$$\begin{aligned} \mathrm{Div}\, {}_\alpha\boldsymbol{P}^* + {}_0\rho\, {}_\alpha\boldsymbol{b}^* &= {}_0\rho\, {}_\alpha\dot{\boldsymbol{v}}^*, \\ \mathrm{Div}\, {}_\alpha\boldsymbol{P}^* &= {}_\alpha\boldsymbol{R}^T(Y, t)\, \mathrm{Div}\, {}_\alpha\boldsymbol{P} \end{aligned} \tag{4.36}$$

where the second of (4.36) is a special case of (2.32) with ${}_\alpha\boldsymbol{Q}(t) = {}_\alpha\boldsymbol{R}^T(Y, t)$.

We next introduce the displacement field of the configuration ${}_2\boldsymbol{\kappa}^*$ relative to ${}_1\boldsymbol{\kappa}^*$, namely

$$ \boldsymbol{h}^* = {}_2\boldsymbol{\chi}^* - {}_1\boldsymbol{\chi}^*, \tag{4.37}$$

so that

$$ \boldsymbol{h}^*(X, t^*) = {}_2\boldsymbol{x}^* - {}_1\boldsymbol{x}^*. \tag{4.38}$$

For a fixed value of t^*, $\boldsymbol{h}^*$ in (4.37) can be expressed as a different function of ${}_1\boldsymbol{x}^*$, t^* in the form

$$\begin{aligned} \boldsymbol{h}^*(X, t) &= (\boldsymbol{h}^* o ({}_1\boldsymbol{\chi}^*)^{-1})({}_1\boldsymbol{x}^*, t^*) \\ &= \{{}_2\boldsymbol{\chi}^* o ({}_1\boldsymbol{\chi}^*)^{-1} - {}_1\boldsymbol{\chi}^* o ({}_1\boldsymbol{\chi}^*)^{-1}\}({}_1\boldsymbol{x}^*, t^*) \end{aligned} \tag{4.39}$$

where (4.37) has been used. Further, at each time t^* a function $\boldsymbol{h}$ may be defined by

$$ \boldsymbol{h} = \boldsymbol{h}^* o ({}_1\boldsymbol{\chi}^*)^{-1}, \tag{4.40}$$

and after recalling $(4.3)_2$, from (4.38) and (4.39) it follows

$$ {}_2\boldsymbol{x}^* - {}_1\boldsymbol{x}^* = \boldsymbol{h}({}_1\boldsymbol{x}^*, t^*) = (\boldsymbol{\chi}^{*\prime} - {}_0\bar{\boldsymbol{\chi}})({}_1\boldsymbol{x}^*, t^*), \tag{4.41}$$

where ${}_0\bar{\boldsymbol{\chi}}$ is the identify mapping on the region ${}_1\mathscr{R}^*$. We note that in view of (4.21),

$$ \boldsymbol{h}({}_1\boldsymbol{y}^*, t^*) = \boldsymbol{o}. \tag{4.42}$$

The relative displacement gradient given by

$$ \boldsymbol{H} = \frac{\partial \boldsymbol{h}}{\partial_1 \boldsymbol{x}^*}({}_1\boldsymbol{x}^*, t^*) \tag{4.43}$$

may, with the aid of (4.41) and (4.9), be expressed as

$$H = {}_2F^*({}_1F^*)^{-1} - I = F^{*\prime} - I, \tag{4.44}$$

the gradient of ${}_0\bar{\chi}$ being the identity tensor. It is readily seen from (4.40), (4.43) and (4.44) that

$$\frac{\partial h^*}{\partial X}(X, t^*) = H\,{}_1F^* = {}_2F^* - {}_1F^*. \tag{4.45}$$

It follows from $(2.17)_1$ and (4.8) that

$$_2n = \frac{((F')^{-1})^T{}_1n}{\|((F')^{-1})^T{}_1n\|}. \tag{4.46}$$

Furthermore, as in $(2.17)_1$,

$$_\alpha n = \frac{((_\alpha F^*)^{-1})^T{}_0n}{\|((_\alpha F^*)^{-1})^T{}_0n\|} \tag{4.47}$$

in the motion ${}_\alpha\chi^*$, and hence in view of (4.9)

$$_2n^* = \frac{((F^{*\prime})^{-1})^T{}_1n^*}{\|((F^{*\prime})^{-1})^T{}_1n^*\|}. \tag{4.48}$$

The restriction (3.13) obtained from the work assumption of Naghdi and Trapp[16] when written for the motion ${}_\alpha\chi^*$ becomes

$$\left(\frac{\partial \bar{S}}{\partial^p E^*}({}_\alpha\mathscr{U}^*) + \frac{\partial \bar{S}}{\partial \kappa^*}({}_\alpha\mathscr{U}^*)\otimes\mathscr{C}({}_\alpha\mathscr{U}^*)\right)[\rho({}_\alpha\mathscr{U}^*)] = -\gamma({}_\alpha\mathscr{U}^*)\frac{\partial g}{\partial E^*}({}_\alpha\mathscr{U}^*). \tag{4.49}$$

It follows from (4.16) and $(4.28)_2$ that (4.49) reduces exactly to (3.13) so that no new restriction is obtained by applying (3.13) to the motion ${}_\alpha\chi^*$.

Let us now apply the construction (4.2) to a motion ${}_\alpha\chi^+$ which differs from ${}_\alpha\chi$ by a rigid motion and thus satisfies an equation of the form (2.7). As in (4.2), we construct a motion $({}_\alpha\chi^+)^* = \pi({}_\alpha\chi^+)$ through the relations

$$({}_\alpha\chi^+)^*(X, t) = \{{}_\alpha R^+(Y, {}_\alpha t^+)\}^T\{{}_\alpha\chi^+(X, {}_\alpha t^+) - {}_\alpha\chi^+(Y, {}_\alpha t^+)\} + Y, \quad t = {}_\alpha t^+ - {}_\alpha a \tag{4.50}$$

where the choice $(4.50)_2$ is made for convenience. The configuration of $\mathscr{B}$ in the motion $({}_\alpha\chi^+)^*$ is denoted by ${}_\alpha\kappa^{+*} = ({}_\alpha\chi^+)^* o_0\kappa$ and the value $({}_\alpha\chi^+)^*(X, t)$ is denoted by $({}_\alpha x^+)^*$.

With the aid of $(2.10)_4$, $(2.7)_1$ and (4.2), from (4.50) we obtain

$$({}_\alpha x^+)^* = {}_\alpha x^*. \tag{4.51}$$

It is of interest to state the relations (4.50) and (4.51) in terms of the functions ω and π as follows:

$$({}_\alpha\chi^+)^* = \pi({}_\alpha\chi^+) = (\pi(\omega({}_\alpha\chi)) = \pi({}_\alpha\chi) = {}_\alpha\chi^*. \tag{4.52}$$

Thus, when a rigid body motion is superposed upon a given motion ${}_\alpha\chi$, resulting in a motion ${}_\alpha\chi^+$ given by (2.7), by applying ω to ${}_\alpha\chi^+$ we arrive at a motion $({}_\alpha\chi^+)^*$ which is equal to ${}_\alpha\chi^*$.

The mapping π extracts from $\mathscr{M}$, the set of all motions of $\mathscr{B}$, a proper subset

$$\mathscr{N} = \{\pi(\theta)|\theta\in\mathscr{M}\} = \pi(\mathscr{M}). \tag{4.53}$$

The notion of invariance under superposed rigid body motions implies that the physical response of a body in the entire set of motions $\mathcal{M}$ is completely determined by its response in the subset $\mathcal{N} \subset \mathcal{M}$.

The image of ${}_0\mathcal{R}$ in the motion $({}_\alpha\chi^+)^*$ is denoted by ${}_\alpha\mathcal{R}^{+*} = ({}_\alpha\chi^+)^*({}_0\mathcal{R}, t)$ and its boundary is denoted by $\partial_\alpha\mathcal{R}^{+*}$. It follows from (4.52) that ${}_\alpha\mathcal{R}^{+*} = {}_\alpha\mathcal{R}^*$, $\partial_\alpha\mathcal{R}^{+*} = \partial_\alpha\mathcal{R}^*$.

In subsequent developments we need to have available explicit relationships between various kinematical quantities calculated from the motions ${}_\alpha\chi^*$ and $({}_\alpha\chi^+)^*$. With the notations

$$ {}_\alpha F^{+*} = \frac{\partial({}_\alpha\chi^+)^*}{\partial X}(X, t), \quad {}_\alpha u^{+*} = (({}_\alpha\chi^+)^* - {}_0\chi)(X, t), \tag{4.54} $$

along with relations paralleling $(4.5)_3$, (4.6) and (4.11), it follows at once from (4.52) that

$$ \begin{gathered} {}_\alpha F^{+*} = {}_\alpha F^*, \quad {}_\alpha u^{+*} = {}_\alpha u^*, \quad {}_\alpha G^{+*} = {}_\alpha G^*, \\ {}_\alpha U^{+*} = {}_\alpha U^*, \quad {}_\alpha C^{+*} = {}_\alpha C^*, \quad {}_\alpha E^{+*} = {}_\alpha E^*, \\ {}_\alpha R^{+*} = {}_\alpha R^*, \end{gathered} \tag{4.55} $$

and

$$ {}_\alpha\dot{F}^{+*} = {}_\alpha\dot{F}^*, \quad {}_\alpha\dot{C}^{+*} = {}_\alpha\dot{C}^*, \quad {}_\alpha\dot{E}^{+*} = {}_\alpha\dot{E}^*. \tag{4.56} $$

We note that in the notation of $(4.54)_1$, ${}_\alpha F^{+*}$ denotes the gradient of the motion $({}_\alpha\chi^+)^* = \pi({}_\alpha\chi^+)$, while ${}_\alpha F^{*+} = ({}_\alpha F^*)^+$ in keeping with $(2.7)_1$ stands for the gradient of a motion $({}_\alpha\chi^*)^+ = \omega({}_\alpha\chi^*)$ which differs from ${}_\alpha\chi^*$ by a superposed rigid body motion. The significance of the results (4.55) lies in the fact that while motions which differ from each other by a rigid motion, and thereby belong to the same equivalence class, in general have different values of G, for example, these motions have the same values of G^*.

It is worth making an observation here for the special case of rigid motions. It follows directly from (2.7) and (4.2) that

$$ \pi({}_0\chi^+) = {}_0\chi \tag{4.57} $$

so that the entire equivalence class of rigid motions is mapped into the identity motion ${}_0\chi$. Consequently, the values of F^*, R^*, C^*, U^*, E^* and G^* in any rigid motion coincide with the values of these fields in the identity motion ${}_0\chi$.

It follows from (4.52) and (3.1) that

$$ {}_\alpha\mathcal{U}^{+*} = {}_\alpha\mathcal{U}^*, \quad {}_\alpha\dot{\mathcal{U}}^{+*} = {}_\alpha\dot{\mathcal{U}}^* \tag{4.58} $$

where ${}_\alpha\mathcal{U}^{+*} = ({}_\alpha E^{+*}, {}^p_\alpha E^{+*}, {}_\alpha\kappa^{+*})$ denotes the value of ${}_\alpha\mathcal{U}$ in the motion $({}_\alpha\chi^+)^*$.

As was previously done for the motion ${}_\alpha\chi^*$ we may associate with the motion $({}_\alpha\chi^+)^*$ quantities ${}_\alpha J^{+*}$, ${}_\alpha\rho^{+*}$, ${}_\alpha b^{+*}$, ${}_\alpha\dot{v}^{+*}$, ${}_\alpha n^{+*}$, ${}_\alpha t^{+*}$, ${}_\alpha T^{+*}$, ${}_\alpha P^{+*}$, ${}_\alpha p^{+*}$, ${}_\alpha S^{+*}$ and the operator ${}_\alpha\mathrm{div}^{+*}$. Then, remembering $(4.55)_1$ and the conservation of mass, it follows that

$$ {}_\alpha J^{+*} = {}_\alpha J^*, \quad {}_\alpha\rho^{+*} = {}_\alpha\rho^*. \tag{4.59} $$

Also, with the help of (4.25), (2.21) and $(2.10)_4$ we obtain

$$ {}_\alpha T^{+*} = \{{}_\alpha Q(t)_\alpha R(Y, t)\}^T{}_\alpha Q(t)_\alpha T_\alpha Q^T(t)_\alpha Q(t)_\alpha R(Y, t) = {}_\alpha T^*. \tag{4.60} $$

Similarly

$$ \begin{gathered} {}_\alpha n^{+*} = {}_\alpha n^*, \quad {}_\alpha t^{+*} = {}_\alpha t^*, \\ {}_\alpha P^{+*} = {}_\alpha P^*, \quad {}_\alpha p^{+*} = {}_\alpha p^*, \\ {}_\alpha S^{+*} = {}_\alpha S^*, \quad {}_\alpha\dot{S}^{+*} = {}_\alpha\dot{S}^*, \\ {}_\alpha\mathrm{div}^{+*}\,{}_\alpha T^{+*} = {}_\alpha\mathrm{div}^*\,{}_\alpha T^*, \\ {}_\alpha v^{+*} = {}_\alpha v^*, \quad {}_\alpha\dot{v}^{+*} = {}_\alpha\dot{v}^*, \quad {}_\alpha b^{+*} = {}_\alpha b^*. \end{gathered} \tag{4.61} $$

Next, as in $(4.3)_1$ we introduce a "difference" motion

$$\chi^{+\prime} = {}_2\chi^{+} o ({}_1\chi^{+})^{-1} \tag{4.62}$$

whose gradient with respect to ${}_1x^{+}$ will be denoted by $F^{+\prime}$. It follows from (4.62), $(2.9)_1$ and (4.8) that

$$F^{+\prime} = {}_2F^{+}({}_1F^{+})^{-1} = {}_2Q(t)F'{}_1Q^{T}(t). \tag{4.63}$$

Similarly, as in $(4.3)_2$ we introduce a "difference" motion

$$\chi^{+*\prime} = ({}_2\chi^{+})^{*} o (({}_1\chi^{+})^{*})^{-1} \tag{4.64}$$

whose gradient with respect to $({}_1x^{+})^{*}$ is denoted by $F^{+*\prime}$. From (4.64), (4.52), $(4.3)_2$ and (4.10) we obtain

$$\chi^{+*\prime} = \chi^{*\prime}, \quad F^{+*\prime} = F^{*\prime}. \tag{4.65}$$

As in (4.37) a displacement field h^{+*} may be defined by

$$h^{+*} = ({}_2\chi^{+})^{*} - ({}_1\chi^{+})^{*} \tag{4.66}$$

so that

$$h^{+*}(X, t) = ({}_2x^{+})^{*} - ({}_1x^{+})^{*}. \tag{4.67}$$

It follows from (4.66), (4.67), (4.52), (4.37) and (4.38) that

$$h^{+*} = h^{*}. \tag{4.68}$$

Now

$$\begin{aligned} h^{+*}(X, t) &= \{h^{+*} o ({}_1\chi^{+*})^{-1}\}(({}_1x^{+})^{*}, t) \\ &= \{({}_2\chi^{+})^{*} o (({}_1\chi^{+})^{*})^{-1} - ({}_1\chi^{+})^{*} o (({}_1\chi^{+})^{*})^{-1}\}(({}_1x^{+})^{*}, t^{*}) \end{aligned} \tag{4.69}$$

where (4.67) has been used. Introducing the composite function

$$h^{+} = h^{+*} o (({}_1\chi^{+})^{*})^{-1} \tag{4.70}$$

and recalling (4.64), from (4.67) and (4.69) we obtain

$$({}_2x^{+})^{*} - ({}_1x^{+})^{*} = h^{+}(({}_1x^{+})^{*}, t) = (\chi^{+*\prime} - {}_0\bar{\chi}^{+})(({}_1x^{+})^{*}, t) \tag{4.71}$$

where ${}_0\bar{\chi}^{+}$ denotes the identity mapping on the region ${}_1\mathscr{R}^{+*} = ({}_1\chi^{+})^{*}({}_0\mathscr{R}, t)$. Inspection of (4.52), (4.68), (4.70) and (4.66) then leads to

$$h^{+} = h. \tag{4.72}$$

Also in view of (4.52), ${}_0\bar{\chi}^{+} = {}_0\bar{\chi}$. Keeping this in mind it is clear that (4.71) is consistent with (4.72), (4.70), (4.68), (4.65), (4.40) and (4.41).

In view of (4.71), (4.72) and (4.43), the relative displacement gradient H^{+}, given by

$$H^{+} = \frac{\partial h^{+}}{\partial ({}_1x^{+})^{*}}(({}_1x^{+})^{*}, t) = F^{+*\prime} - I, \tag{4.73}$$

satisfies

$$H^{+} = H, \tag{4.74}$$

and hence also

$$\dot{H}^{+} = \dot{H}. \tag{4.75}$$

5. SMALL DEFORMATIONS SUPERPOSED ON A LARGE DEFORMATION

In this section we regard ${}_1\chi$ as an arbitrary known motion of $\mathscr{B}$ and ${}_2\chi$ as a general motion of $\mathscr{B}$ differing from ${}_1\chi$ in such a way that the state of the elastic–plastic material undergoing the motion ${}_2\chi$ is close (in a sense to be made precise) to that of the same material undergoing the motion ${}_1\chi$.

From ${}_1\chi$ and ${}_2\chi$ we construct the associated motions ${}_1\chi^*$ and ${}_2\chi^*$ in accordance with (4.2). The relative displacement between the configurations ${}_2\kappa^*$ and ${}_1\kappa^*$ is then given by (4.37) and its gradient relative to the configuration ${}_1\kappa^*$ is given by (4.43).

The motion ${}_1\chi$, and hence also ${}_1\chi^*$, may be inducing loading, neutral loading, unloading or an elastic state in the elastic–plastic material. The motion ${}_2\chi$, and with it ${}_2\chi^*$, may independently be inducing any of these states also. In order to establish a measure of smallness we introduce the following nonnegative real functions:

$$\epsilon_1 = \epsilon_1(t^*) = \sup_{{}_1x^* \in {}_1\mathscr{R}^*} \|H({}_1x^*, t^*)\|,$$
$$\epsilon_2 = \epsilon_2(t^*) = \sup_{{}_1x^* \in {}_1\mathscr{R}} \|\dot{H}({}_1x^*, t^*)\|, \tag{5.1}$$

$$\begin{aligned}\epsilon_3 = \epsilon_3(t^*) &= \sup_{{}_1x^* \in {}_1\mathscr{R}^*,\ {}_2x^* \in {}_2\mathscr{R}^*} \|{}^p_2E^*({}_2x^*, t^*) - {}^p_1E^*({}_1x^*, t^*)\| \\ &= \sup_{{}_1x \in {}_1\mathscr{R},\ {}_2x \in {}_2\mathscr{R}} \|{}^p_2E({}_2x, t) - {}^p_1E({}_1x, t)\|\end{aligned} \tag{5.2}$$

and

$$\begin{aligned}\epsilon_4 = \epsilon_4(t^*) &= \sup_{{}_1x^* \in {}_1\mathscr{R}^*,\ {}_2x^* \in {}_2\mathscr{R}^*} \|{}_2\kappa^*({}_2x^*, t^*) - {}_1\kappa^*({}_1x^*, t^*)\| \\ &= \sup_{{}_1x \in {}_1\mathscr{R},\ {}_2x \in {}_2\mathscr{R}} \|{}_2\kappa({}_2x, t) - {}_1\kappa({}_1x, t)\|\end{aligned} \tag{5.3}$$

where ${}_\alpha x^*$ are given by (4.2) and where use has been made of (4.16); sup stands for the supremum or least upper bound of a nonempty bounded set of real numbers. As our basic measure of smallness we will employ the function

$$\epsilon = \epsilon(t^*) = \max\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}. \tag{5.4}$$

If Z is a tensor-valued function† of the variables H, ${}^p_1E^*$, ${}^p_2E^*$, ${}_1\kappa^*$, ${}_2\kappa^*$ and t^* defined at time t^* in a neighborhood of $H = 0$, ${}^p_2E^* - {}^p_1E^* = 0$, ${}_2\kappa^* - {}_1\kappa^* = 0$ and satisfying the condition that there exists a nonnegative real constant K such that

$$\|Z\| < K\epsilon^n \text{ as } \epsilon \to 0, \tag{5.5}$$

then we write

$$Z = 0(\epsilon^n) \text{ as } \epsilon \to 0,$$

n being a nonnegative integer.

It follows from (4.44), (4.6), $(5.1)_1$, (5.4) and (5.5) that

$$\begin{aligned}{}_2E^* - {}_1E^* = \tfrac{1}{2}({}_1F^*)^T(H + H^T + H^TH){}_1F^* &= \Delta E^* + 0(\epsilon^2) \text{ as } \epsilon \to 0 \\ &= 0(\epsilon) \text{ as } \epsilon \to 0,\end{aligned} \tag{5.6a}$$

†Z may be a second order, first order (vector), or scalar valued function.

where

$$\Delta \boldsymbol{E}^* = \tfrac{1}{2}({}_1\boldsymbol{F}^*)^T(\boldsymbol{H} + \boldsymbol{H}^T)_1\boldsymbol{F}^* = \mathbf{0}(\epsilon) \text{ as } \epsilon \to 0. \tag{5.6b}$$

Then, taking a material time derivative of both sides of $(5.6a)_1$ and invoking (5.1), (5.4), (5.5) and (5.6b), we obtain

$$\begin{aligned} {}_2\dot{\boldsymbol{E}}^* - {}_1\dot{\boldsymbol{E}}^* &= \tfrac{1}{2}({}_1\dot{\boldsymbol{F}}^*)^T(\boldsymbol{H} + \boldsymbol{H}^T)_1\boldsymbol{F}^* + \tfrac{1}{2}({}_1\boldsymbol{F}^*)^T(\boldsymbol{H} + \boldsymbol{H}^T)_1\dot{\boldsymbol{F}}^* \\ &\quad + \tfrac{1}{2}({}_1\boldsymbol{F}^*)^T(\dot{\boldsymbol{H}} + \dot{\boldsymbol{H}}^T)_1\boldsymbol{F}^* + \overline{\tfrac{1}{2}({}_1\boldsymbol{F}^*)^T\boldsymbol{H}^T\boldsymbol{H}_1\boldsymbol{F}^*}^{\,\boldsymbol{\cdot}} \\ &= \dot{\overline{\Delta \boldsymbol{E}^*}} + \mathbf{0}(\epsilon^2) \text{ as } \epsilon \to 0 \\ &= \mathbf{0}(\epsilon) \text{ as } \epsilon \to 0. \end{aligned} \tag{5.7}$$

We note in passing that Shack ([8] eqn 3.5) does not include the contribution due to the first pair of terms on the r.h.s. of $(5.7)_1$ in his calculation of strain-rate differences. If it is assumed that $\boldsymbol{H}$ (but not $\dot{\boldsymbol{H}}$) is equal to zero at time t, Shack's equation is recovered.

Performing a polar decomposition on $\boldsymbol{F}^{*\prime}$ in (4.9) results in

$$\boldsymbol{F}^{*\prime} = \boldsymbol{R}^{*\prime}\boldsymbol{U}^{*\prime} \tag{5.8}$$

with $\boldsymbol{R}^{*\prime}$ a proper orthogonal tensor, and $\boldsymbol{U}^{*\prime}$ a symmetric positive definite tensor. Then with the aid of (4.4), $(5.1)_1$, (5.4) and (5.5) it is readily shown that

$$\begin{aligned} (\boldsymbol{F}^{*\prime})^{-1} - \boldsymbol{I} &= -\boldsymbol{H} + \mathbf{0}(\epsilon^2) = \mathbf{0}(\epsilon), && \text{(a)} \\ \boldsymbol{U}^{*\prime} - \boldsymbol{I} &= \tfrac{1}{2}(\boldsymbol{H} + \boldsymbol{H}^T) + \mathbf{0}(\epsilon^2) = \mathbf{0}(\epsilon), && \text{(b)} \\ (\boldsymbol{U}^{*\prime})^{-1} - \boldsymbol{I} &= -\tfrac{1}{2}(\boldsymbol{H} + \boldsymbol{H}^T) + \mathbf{0}(\epsilon^2) = \mathbf{0}(\epsilon), && \text{(c)} \\ \boldsymbol{R}^{*\prime} - \boldsymbol{I} &= \tfrac{1}{2}(\boldsymbol{H} - \boldsymbol{H}^T) + \mathbf{0}(\epsilon^2) = \mathbf{0}(\epsilon), && \text{(d)} \\ (\boldsymbol{R}^{*\prime})^T - \boldsymbol{I} &= -\tfrac{1}{2}(\boldsymbol{H} - \boldsymbol{H}^T) + \mathbf{0}(\epsilon^2) = \mathbf{0}(\epsilon), && \text{(e)} \end{aligned} \tag{5.9}$$

as $\epsilon \to 0$.

Assuming sufficient smoothness, we expand $g({}_2\mathscr{U}^*)$ in a Taylor series about the point ${}_1\mathscr{U}^*$ and invoke (5.6), (5.2), (5.3), (5.4) and (5.5) to obtain

$$g({}_2\mathscr{U}^*) - g({}_1\mathscr{U}^*) = \Delta g + 0(\epsilon^2) \text{ as } \epsilon \to 0 \tag{5.10a}$$

where

$$\begin{aligned} \Delta g &= \frac{\partial g}{\partial \boldsymbol{E}^*}({}_1\mathscr{U}^*)\cdot \Delta \boldsymbol{E}^* + \frac{\partial g}{\partial^p \boldsymbol{E}^*}({}_1\mathscr{U}^*)\cdot({}_2^p\boldsymbol{E}^* - {}_1^p\boldsymbol{E}^*) + \frac{\partial g}{\partial \kappa^*}({}_1\mathscr{U}^*)({}_2\kappa^* - {}_1\kappa^*) \\ &= 0(\epsilon) \text{ as } \epsilon \to 0. \end{aligned} \tag{5.10b}$$

Similarly,

$$\frac{\partial g}{\partial \boldsymbol{E}^*}({}_2\mathscr{U}^*) - \frac{\partial g}{\partial \boldsymbol{E}^*}({}_1\mathscr{U}^*) = \Delta\left(\frac{\partial g}{\partial \boldsymbol{E}^*}\right) + \mathbf{0}(\epsilon^2) \text{ as } \epsilon \to 0, \tag{5.11a}$$

where

$$\begin{aligned} \Delta\left(\frac{\partial g}{\partial \boldsymbol{E}^*}\right) &= \frac{\partial^2 g}{\partial \boldsymbol{E}^* \partial \boldsymbol{E}^*}({}_1\mathscr{U}^*)[\Delta \boldsymbol{E}^*] + \frac{\partial^2 g}{\partial \boldsymbol{E}^* \partial^p \boldsymbol{E}^*}({}_1\mathscr{U}^*)[{}_2^p\boldsymbol{E}^* - {}_1^p\boldsymbol{E}^*] + \frac{\partial^2 g}{\partial \boldsymbol{E}^* \partial \kappa^*}({}_1\mathscr{U}^*)({}_2\kappa^* - {}_1\kappa^*) \\ &= \mathbf{0}(\epsilon) \text{ as } \epsilon \to 0. \end{aligned} \tag{5.11b}$$

From (4.19), (5.11) and (5.7) we obtain

$$ {}_2\hat{g}^* - {}_1\hat{g}^* = \xi + 0(\epsilon^2) \text{ as } \epsilon \to 0, \tag{5.12a}$$

where

$$\xi = \frac{\partial g}{\partial \boldsymbol{E}^*}({}_1\mathscr{U}^*)\cdot \overline{\Delta \boldsymbol{E}^*}^{\,\cdot} + \Delta\left(\frac{\partial g}{\partial \boldsymbol{E}^*}\right)\cdot {}_1\dot{\boldsymbol{E}}^* = 0(\epsilon) \text{ as } \epsilon \to 0. \tag{5.12b}$$

In our analysis, we neglect terms of $0(\epsilon^2)$ as $\epsilon \to 0$. The difference between $g({}_2\mathscr{U}^*)$ and $g({}_1\mathscr{U}^*)$, and the difference between ${}_2\hat{g}^*$ and ${}_1\hat{g}^*$ will then be given by the terms of $0(\epsilon)$ in (5.10) and (5.12), respectively. Consequently, the change in loading conditions between the two motions ${}_1\boldsymbol{\chi}^*$ and ${}_2\boldsymbol{\chi}^*$ (and hence also between ${}_1\boldsymbol{\chi}$ and ${}_2\boldsymbol{\chi}$) will at each time t be determined by these terms of $0(\epsilon)$. Thus, omitting terms of $0(\epsilon^2)$ in (5.10a) and (5.12a), we write

$$g({}_2\mathscr{U}^*) = g({}_1\mathscr{U}^*) + \Delta g \tag{5.13}$$

and

$${}_2\hat{g}^* = {}_1\hat{g}^* + \xi. \tag{5.14}$$

It follows from (3.5), $(4.12)_3$, (4.16) and (5.13) that

$$g({}_1\mathscr{U}^*) + \Delta g \leqslant 0. \tag{5.15}$$

Since any of the four cases in (4.20) [or (3.7)] may be occurring in either ${}_2\boldsymbol{\chi}^*$ (and ${}_2\boldsymbol{\chi}$) or ${}_1\boldsymbol{\chi}^*$ (and ${}_1\boldsymbol{\chi}$), we have to consider sixteen distinct cases altogether. For convenience we designate each of these by an ordered couple (C_1, C_2) with $C_1 = E, U, N$ or L corresponding, respectively, to an elastic state, unloading from an elastic–plastic state, neutral loading from an elastic–plastic state and loading from an elastic–plastic state in the motion ${}_1\boldsymbol{\chi}^*$ (and ${}_1\boldsymbol{\chi}$), and $C_2 = E, U, N$ or L corresponding to these states in the motion ${}_2\boldsymbol{\chi}^*$ (and ${}_2\boldsymbol{\chi}$). We shall examine each of the sixteen separate cases presently but first we will need to derive approximations from the eqns (4.31) and (4.26) as well as some related formulae that will be used later.

Expanding $\bar{\boldsymbol{S}}({}_2\mathscr{U}^*)$ about the point ${}_1\mathscr{U}^*$ and invoking (5.6), (5.2), (5.3), (5.4) and (5.5), we obtain

$${}_2\boldsymbol{S}^* - {}_1\boldsymbol{S}^* = \Delta \boldsymbol{S}^* + 0(\epsilon^2) \text{ as } \epsilon \to 0 \tag{5.16a}$$

where

$$\begin{aligned}\Delta \boldsymbol{S}^* &= \frac{\partial \bar{\boldsymbol{S}}}{\partial \boldsymbol{E}^*}({}_1\mathscr{U}^*)[\Delta \boldsymbol{E}^*] + \frac{\partial \bar{\boldsymbol{S}}}{\partial^p \boldsymbol{E}^*}({}_1\mathscr{U}^*)[{}^p_2\boldsymbol{E}^* - {}^p_1\boldsymbol{E}^*] + \frac{\partial \bar{\boldsymbol{S}}}{\partial \kappa^*}({}_1\mathscr{U}^*)({}_2\kappa^* - {}_1\kappa^*)\\ &= 0(\epsilon) \text{ as } \epsilon \to 0.\end{aligned} \tag{5.16b}$$

Similarly

$$\frac{\partial \bar{\boldsymbol{S}}}{\partial^p \boldsymbol{E}^*}({}_2\mathscr{U}^*) - \frac{\partial \bar{\boldsymbol{S}}}{\partial^p \boldsymbol{E}^*}({}_1\mathscr{U}^*) = \Delta\left(\frac{\partial \bar{\boldsymbol{S}}}{\partial^p \boldsymbol{E}^*}\right) + 0(\epsilon^2) \text{ as } \epsilon \to 0 \tag{5.17a}$$

where

$$\begin{aligned}\Delta\left(\frac{\partial \bar{\boldsymbol{S}}}{\partial^p \boldsymbol{E}^*}\right) &= \frac{\partial^2 \bar{\boldsymbol{S}}}{\partial^p \boldsymbol{E}^* \partial \boldsymbol{E}^*}({}_1\mathscr{U}^*)[\Delta \boldsymbol{E}^*] + \frac{\partial^2 \bar{\boldsymbol{S}}}{\partial^p \boldsymbol{E}^* \partial^p \boldsymbol{E}^*}({}_1\mathscr{U}^*)[{}^p_2\boldsymbol{E}^* - {}^p_1\boldsymbol{E}^*]\\ &\quad + \frac{\partial^2 \bar{\boldsymbol{S}}}{\partial^p \boldsymbol{E}^* \partial \kappa^*}({}_1\mathscr{U}^*)({}_2\kappa^* - {}_1\kappa^*)\\ &= 0(\epsilon) \text{ as } \epsilon \to 0.\end{aligned} \tag{5.17b}$$

Also

$$\frac{\partial \bar{\boldsymbol{S}}}{\partial \kappa^*}({}_2\mathscr{U}^*) - \frac{\partial \bar{\boldsymbol{S}}}{\partial \kappa^*}({}_1\mathscr{U}^*) = \Delta\left(\frac{\partial \bar{\boldsymbol{S}}}{\partial \kappa^*}\right) + \mathbf{0}(\epsilon^2) \text{ as } \epsilon \to 0 \tag{5.18a}$$

where

$$\Delta\left(\frac{\partial \bar{\boldsymbol{S}}}{\partial \kappa^*}\right) = \frac{\partial^2 \bar{\boldsymbol{S}}}{\partial \kappa^* \partial \boldsymbol{E}^*}({}_1\mathscr{U}^*)[\Delta \boldsymbol{E}^*] + \frac{\partial^2 \bar{\boldsymbol{S}}}{\partial \kappa^* \partial^p \boldsymbol{E}^*}({}_1\mathscr{U}^*)[{}^p_2\boldsymbol{E}^* - {}^p_1\boldsymbol{E}^*) + \frac{\partial^2 \bar{\boldsymbol{S}}}{\partial (\kappa^*)^2}({}_1\mathscr{U}^*)({}_2\kappa^* - {}_1\kappa^*)$$
$$= \mathbf{0}(\epsilon) \text{ as } \epsilon \to 0. \tag{5.18b}$$

Furthermore,

$$\boldsymbol{\rho}({}_2\mathscr{U}^*) - \boldsymbol{\rho}({}_1\mathscr{U}^*) = \Delta \boldsymbol{\rho} + \mathbf{0}(\epsilon^2) \text{ as } \epsilon \to 0, \tag{5.19a}$$

where

$$\Delta \boldsymbol{\rho} = \frac{\partial \boldsymbol{\rho}}{\partial \boldsymbol{E}^*}({}_1\mathscr{U}^*)[\Delta \boldsymbol{E}^*] + \frac{\partial \boldsymbol{\rho}}{\partial^p \boldsymbol{E}^*}({}_1\mathscr{U}^*)[{}^p_2\boldsymbol{E}^* - {}^p_1\boldsymbol{E}^*] + \frac{\partial \boldsymbol{\rho}}{\partial \kappa^*}({}_1\mathscr{U}^*)({}_2\kappa^* - {}_1\kappa^*)$$
$$= \mathbf{0}(\epsilon) \text{ as } \epsilon \to 0. \tag{5.19b}$$

Similarly,

$$\lambda({}_2\mathscr{U}^*) - \lambda({}_1\mathscr{U}^*) = \Delta\lambda + 0(\epsilon^2) \text{ as } \epsilon \to 0, \tag{5.20a}$$

where

$$\Delta\lambda = \frac{\partial \lambda}{\partial \boldsymbol{E}^*}({}_1\mathscr{U}^*) \cdot \Delta \boldsymbol{E}^* + \frac{\partial \lambda}{\partial^p \boldsymbol{E}^*}({}_1\mathscr{U}^*) \cdot ({}^p_2\boldsymbol{E}^* - {}^p_1\boldsymbol{E}^*) + \frac{\partial \lambda}{\partial \kappa^*}({}_1\mathscr{U}^*)({}_2\kappa^* - {}_1\kappa^*)$$
$$= \mathbf{0}(\epsilon) \text{ as } \epsilon \to 0. \tag{5.20b}$$

Also

$$\mathscr{C}({}_2\mathscr{U}^*) - \mathscr{C}({}_1\mathscr{U}^*) = \Delta\mathscr{C} + \mathbf{0}(\epsilon^2) \text{ as } \epsilon \to 0 \tag{5.21a}$$

where

$$\Delta\mathscr{C} = \frac{\partial \mathscr{C}}{\partial \boldsymbol{E}^*}({}_1\mathscr{U}^*)[\Delta \boldsymbol{E}^*] + \frac{\partial \mathscr{C}}{\partial^p \boldsymbol{E}^*}({}_1\mathscr{U}^*)[{}^p_2\boldsymbol{E}^* - {}^p_1\boldsymbol{E}^*] + \frac{\partial \mathscr{C}}{\partial \kappa^*}({}_1\mathscr{U}^*)({}_2\kappa^* - {}_1\kappa^*)$$
$$= \mathbf{0}(\epsilon) \text{ as } \epsilon \to 0. \tag{5.21b}$$

Finally, we also need the relation

$$\frac{\partial g}{\partial^p \boldsymbol{E}^*}({}_2\mathscr{U}^*) - \frac{\partial g}{\partial^p \boldsymbol{E}^*}({}_1\mathscr{U}^*) = \Delta\left(\frac{\partial g}{\partial^p \boldsymbol{E}^*}\right) + \mathbf{0}(\epsilon^2) \text{ as } \epsilon \to 0 \tag{5.22a}$$

where

$$\Delta\left(\frac{\partial g}{\partial^p \boldsymbol{E}^*}\right) = \frac{\partial^2 g}{\partial^p \boldsymbol{E}^* \partial \boldsymbol{E}^*}({}_1\mathscr{U}^*)[\Delta \boldsymbol{E}^*] + \frac{\partial^2 g}{\partial^p \boldsymbol{E}^* \partial^p \boldsymbol{E}^*}({}_1\mathscr{U}^*)[{}^p_2\boldsymbol{E}^* - {}^p_1\boldsymbol{E}^*] + \frac{\partial^2 g}{\partial^p \boldsymbol{E}^* \partial \kappa^*}({}_1\mathscr{U}^*)({}_2\kappa^* - {}_1\kappa^*)$$
$$= \mathbf{0}(\epsilon) \text{ as } \epsilon \to 0, \tag{5.22b}$$

as well as the relation

$$\frac{\partial g}{\partial \kappa^*}({}_2\mathscr{U}^*) - \frac{\partial g}{\partial \kappa^*}({}_1\mathscr{U}^*) = \Delta\left(\frac{\partial g}{\partial \kappa^*}\right) + 0(\epsilon^2) \text{ as } \epsilon \to 0 \tag{5.23a}$$

where

$$\Delta\left(\frac{\partial g}{\partial \kappa^*}\right) = \frac{\partial^2 g}{\partial \kappa^* \partial \mathbf{E}^*}({}_1\mathscr{U}^*)\cdot \Delta \mathbf{E}^* + \frac{\partial^2 g}{\partial \kappa^* \partial^p \mathbf{E}^*}({}_1\mathscr{U}^*)\cdot ({}_2^p\mathbf{E}^* - {}_1^p\mathbf{E}^*) + \frac{\partial^2 g}{\partial (\kappa^*)^2}({}_1\mathscr{U}^*)({}_2\kappa^* - {}_1\kappa^*)$$

$$= 0(\epsilon) \text{ as } \epsilon \to 0. \tag{5.23b}$$

With the help of (5.19)–(5.23), it follows from (3.12), evaluated at ${}_1\mathscr{U}^*$ and ${}_2\mathscr{U}^*$, that

$$\lambda({}_1\mathscr{U}^*)\boldsymbol{\rho}({}_1\mathscr{U}^*)\cdot\left\{\Delta\left(\frac{\partial g}{\partial^p \mathbf{E}^*}\right) + \frac{\partial g}{\partial \kappa^*}({}_1\mathscr{U}^*)\Delta\mathscr{C} + \mathscr{C}({}_1\mathscr{U}^*)\Delta\left(\frac{\partial g}{\partial \kappa^*}\right)\right\}$$
$$+ \left\{\frac{\partial g}{\partial^p \mathbf{E}^*}({}_1\mathscr{U}^*) + \frac{\partial g}{\partial \kappa^*}({}_1\mathscr{U}^*)\mathscr{C}({}_1\mathscr{U}^*)\right\}\cdot\left\{\lambda({}_1\mathscr{U}^*)\Delta\boldsymbol{\rho}\right.$$
$$\left. + \boldsymbol{\rho}({}_1\mathscr{U}^*)\Delta\lambda\right\} = 0 + 0(\epsilon^2) \text{ as } \epsilon \to 0, \tag{5.24a}$$

which may also be written as

$$\boldsymbol{\rho}({}_1\mathscr{U}^*)\cdot\left\{\Delta\left(\frac{\partial g}{\partial^p \mathbf{E}^*}\right) + \frac{\partial g}{\partial \kappa^*}({}_1\mathscr{U}^*)\Delta\mathscr{C} + \mathscr{C}({}_1\mathscr{U}^*)\Delta\left(\frac{\partial g}{\partial \kappa^*}\right)\right\} + \left\{\frac{\partial g}{\partial^p \mathbf{E}^*}({}_1\mathscr{U}^*) + \frac{\partial g}{\partial \kappa^*}({}_1\mathscr{U}^*)\mathscr{C}({}_1\mathscr{U}^*)\right\}$$
$$\cdot\Delta\boldsymbol{\rho} - \frac{\Delta\lambda}{\lambda^2({}_1\mathscr{U}^*)} = 0 + 0(\epsilon^2) \text{ as } \epsilon \to 0. \tag{5.24b}$$

Suppose now that the motion ${}_2\boldsymbol{\chi}^*$ is inducing loading from an elastic–plastic state. Then, by (4.20d), (4.21b), (5.13), (5.14), (5.19)–(5.21)

$$g({}_1\mathscr{U}^*) + \Delta g = 0, \quad {}_1\hat{g}^* + \xi > 0 \tag{5.25a}$$

and

$${}_2^p\dot{\mathbf{E}}^* = \lambda({}_2\mathscr{U}^*){}_2\hat{g}^*\boldsymbol{\rho}({}_2\mathscr{U}^*) = \{\lambda({}_1\mathscr{U}^*)(\boldsymbol{\rho}({}_1\mathscr{U}^*) + \Delta\boldsymbol{\rho}) + \boldsymbol{\rho}({}_1\mathscr{U}^*)\Delta\lambda\}{}_1\hat{g}^* + \lambda({}_1\mathscr{U}^*)\boldsymbol{\rho}({}_1\mathscr{U}^*)\xi \tag{5.25b}$$

while

$${}_2\dot{\kappa} = \{\lambda({}_1\mathscr{U}^*)(\mathscr{C}({}_1\mathscr{U}^*) + \Delta\mathscr{C})\cdot\boldsymbol{\rho}({}_1\mathscr{U}^*) + \mathscr{C}({}_1\mathscr{U}^*)\cdot(\lambda({}_1\mathscr{U}^*)\Delta\boldsymbol{\rho} + \boldsymbol{\rho}({}_1\mathscr{U}^*)\Delta\lambda)\}{}_1\hat{g}^*$$
$$+ \lambda({}_1\mathscr{U}^*)\mathscr{C}({}_1\mathscr{U}^*)\cdot\boldsymbol{\rho}({}_1\mathscr{U}^*)\xi, \tag{5.25c}$$

where terms of $0(\epsilon^2)$ as $\epsilon \to 0$ have been omitted.

Equations (5.25) hold regardless of the condition of loading that is occurring in the motion ${}_1\boldsymbol{\chi}^*$. If, however, loading is being induced in the motion ${}_1\boldsymbol{\chi}^*$ as well as in the motion ${}_2\boldsymbol{\chi}^*$, then from (5.25), (4.20d) and (4.21b) we obtain

$$g({}_1\mathscr{U}^*) = 0, \ \Delta g = 0,$$
$${}_1\hat{g}^* > 0, \ {}_1\hat{g}^* + \xi > 0, \tag{5.26a}$$

$${}_2^p\dot{\mathbf{E}}^* = {}_1^p\dot{\mathbf{E}}^* + \{\lambda({}_1\mathscr{U}^*)\Delta\boldsymbol{\rho} + \boldsymbol{\rho}({}_1\mathscr{U}^*)\Delta\lambda\}{}_1\hat{g}^* + \lambda({}_1\mathscr{U}^*)\boldsymbol{\rho}({}_1\mathscr{U}^*)\xi, \tag{5.26b}$$

and

$${}_2\dot{\kappa}^* = {}_1\dot{\kappa}^* + \{\lambda({}_1\mathscr{U}^*)\boldsymbol{\rho}({}_1\mathscr{U}^*)\cdot\Delta\mathscr{C} + \mathscr{C}({}_1\mathscr{U}^*)\cdot(\lambda({}_1\mathscr{U}^*)\Delta\boldsymbol{\rho} + \boldsymbol{\rho}({}_1\mathscr{U}^*)\Delta\lambda)\}{}_1\hat{g}^*$$
$$+ \lambda({}_1\mathscr{U}^*)\mathscr{C}({}_1\mathscr{U}^*)\cdot\boldsymbol{\rho}({}_1\mathscr{U}^*)\xi. \tag{5.26c}$$

We now examine the sixteen separate cases mentioned above.

Case I: (E, E)

Since ${}_1\chi^*$ is inducing an elastic state, then by (4.20a) and (4.21a)

$$g({}_1\mathscr{U}^*) < 0, \quad {}^p_1\dot{\boldsymbol{E}}^* = \boldsymbol{0}, \quad {}_1\dot{\kappa}^* = 0. \tag{5.27}$$

In the context of the approximate theory being developed it follows from (4.20a) and (5.13) that a necessary and sufficient condition for an elastic state in the motion ${}_2\chi^*$ (and ${}_2\chi$) is that

$$g({}_1\mathscr{U}^*) + \Delta g < 0. \tag{5.28}$$

When (5.28) holds, then in view of (4.20a) and (4.21a)

$${}^p_2\dot{\boldsymbol{E}}^* = \boldsymbol{0}, \quad {}_2\dot{\kappa}^* = 0. \tag{5.29}$$

Case II: (E, U)

In this case (5.27) still hold. It follows from (4.20b), (5.13) and (5.14) that necessary and sufficient conditions for unloading from an elastic–plastic state in the motion ${}_2\chi^*$ are that

$$g({}_1\mathscr{U}^*) + \Delta g = 0, \quad {}_1\hat{g}^* + \xi < 0. \tag{5.30}$$

It then follows from (4.20b) and (4.21a) that (5.29) still hold.

Case III: (E, N)

Again (5.27) hold. It follows from (4.20c), (5.13) and (5.14) that necessary and sufficient conditions for neutral loading in the motion ${}_2\chi^*$ are that

$$g({}_1\mathscr{U}^*) + \Delta g = 0, \quad {}_1\hat{g}^* + \xi = 0. \tag{5.31}$$

Then, by (4.20c) and (4.21a), we see that (5.29) still hold. Consistent with this is the fact that (5.25b,c) vanish in this case (to within terms of $\mathbf{0}(\epsilon^2)$ as $\epsilon \to 0$) due to the presence of the factor (5.14).

Case IV: (E, L)

For this case also (5.27) hold. In view of (4.20d), (5.13) and (5.14) necessary and sufficient conditions for loading in the motion ${}_2\chi^*$ are that

$$g({}_1\mathscr{U}^*) + \Delta g = 0, \quad {}_1\hat{g}^* + \xi > 0. \tag{5.32}$$

${}^p_2\dot{\boldsymbol{E}}^*$ and ${}_2\dot{\kappa}$ are then given by (5.25b) and (5.25c), respectively.

Case V: (U, E)

For this case it follows from (4.20b) and (4.21a) that

$$\begin{gathered} g({}_1\mathscr{U}^*) = 0, \quad {}_1\hat{g}^* < 0, \\ {}^p_1\dot{\boldsymbol{E}}^* = \boldsymbol{0}, \quad {}_1\dot{\kappa}^* = 0. \end{gathered} \tag{5.33}$$

In view of (5.33), (4.20) and (5.13) a necessary and sufficient condition for an elastic state in the motion ${}_2\chi^*$ is that

$$\Delta g < 0. \tag{5.34}$$

With (5.34) holding, it follows from (4.20a) and (4.21a) that ${}^p_2\dot{\boldsymbol{E}}^*$ and ${}_2\dot{\kappa}^*$ are given by (5.29).

Case VI: (U, U)

Again (5.33) hold. It follows from $(5.33)_1$, (4.20b), (5.13) and (5.14) that necessary and sufficient conditions for unloading in the motion ${}_2\chi^*$ are that

$$\Delta g = 0, \quad {}_1\hat{g}^* + \xi < 0. \tag{5.35}$$

Then ${}^p_2\dot{\mathbf{E}}^*$ and ${}_2\dot{\kappa}^*$ are given by (5.29).

Case VII: (U, N)

In this case also (5.33) hold. Necessary and sufficient conditions for neutral loading in the motion ${}_2\chi^*$ are that

$$\Delta g = 0, \quad {}_1\hat{g}^* + \xi = 0 \tag{5.36}$$

where $(5.33)_1$, (4.20c), (5.13) and (5.14) have been used. Again ${}^p_2\dot{\mathbf{E}}^*$ and ${}_2\dot{\kappa}^*$ are given by (5.29). We note that (5.25b,c) vanish in this case.

Case VIII: (U, L)

The relations (5.33) hold. In view of $(5.33)_1$, (4.20d), (5.13) and (5.14), necessary and sufficient conditions for loading in the motion ${}_2\chi^*$ are that

$$\Delta g = 0, \quad {}_1\hat{g}^* + \xi > 0. \tag{5.37}$$

Then ${}^p_2\dot{\mathbf{E}}^*$ and ${}_2\dot{\kappa}^*$ are given by (5.25b) and (5.25c), respectively.

Case IX: (N, E)

For this case, it follows from (4.20c) and (4.21a) that

$$g({}_1\mathscr{U}^*) = 0, \quad {}_1\hat{g}^* = 0, \quad {}^p_1\dot{\mathbf{E}}^* = \mathbf{0},. \quad {}_1\dot{\kappa}^* = 0. \tag{5.38}$$

Then in view of $(5.38)_1$, (4.20a) and (5.13), the inequality (5.34) provides a necessary and sufficient condition for an elastic state in the motion ${}_2\chi^*$. ${}^p_2\dot{\mathbf{E}}^*$ and ${}_2\dot{\kappa}^*$ are given by (5.29).

Case X: (N, U)

Again (5.38) hold. Necessary and sufficient conditions for unloading in ${}_2\chi^*$ are that

$$\Delta g = 0; \quad \xi < 0, \tag{5.39}$$

where (4.20b), (5.13), (5.14) and $(5.38)_{1,2}$ have been used. ${}^p_2\dot{\mathbf{E}}^*$ and ${}_2\dot{\kappa}^*$ are again given by (5.29).

Case XI: (N, N)

The relations (5.38) hold. Necessary and sufficient conditions for neutral loading in ${}_2\chi^*$ are that

$$\Delta g = 0, \quad \xi = 0, \tag{5.40}$$

where (4.20b), (5.13), (5.14) and $(5.38)_{1,2}$ have been used. Again (5.29) hold.

Case XII: (N, L)

Equations (5.38) hold. It follows from (4.20d), (5.13), (5.14) and $(5.38)_{1,2}$ that necessary and sufficient conditions for loading in the motion ${}_2\chi^*$ are that

$$\Delta g = 0, \quad \xi > 0. \tag{5.41}$$

From (5.25b,c) and $(5.38)_2$ we obtain

$$\begin{aligned} {}^p_2\dot{\mathbf{E}}^* &= \lambda({}_1\mathscr{U}^*)\xi\boldsymbol{\rho}({}_1\mathscr{U}^*), \\ {}_2\dot{\kappa}^* &= \lambda({}_1\mathscr{U}^*)\xi\mathscr{C}({}_1\mathscr{U}^*)\cdot\boldsymbol{\rho}({}_1\mathscr{U}^*). \end{aligned} \tag{5.42}$$

Case XIII: (L, E)

In this case it follows from (4.20d) and (4.21b) that

$$g({}_1\mathcal{U}^*) = 0, \quad {}_1\hat{g}^* > 0,$$
$${}^p_1\dot{\boldsymbol{E}}^* = \lambda({}_1\mathcal{U}^*){}_1\hat{g}^*\boldsymbol{\rho}({}_1\mathcal{U}^*), \tag{5.43}$$
$${}_1\dot{\kappa}^* = \lambda({}_1\mathcal{U}^*){}_1\hat{g}^*\mathscr{C}({}_1\mathcal{U}^*)\cdot\boldsymbol{\rho}({}_1\mathcal{U}^*).$$

In view of $(5.43)_1$, (4.20a) and (5.13), (5.34) is a necessary and sufficient condition for an elastic state in the motion ${}_2\boldsymbol{\chi}^*$. Then ${}^p_2\dot{\boldsymbol{E}}^*$ and ${}_2\dot{\kappa}^*$ are given by (5.29).

Case XIV: (L, U)

The relations (5.43) hold. It follows from (4.20b), (5.13), (5.14) and $(5.43)_1$ that necessary and sufficient conditions for unloading in the motion ${}_2\boldsymbol{\chi}^*$ are given by (5.35). Equations (5.29) still hold.

Case XV: (L, N)

Again (5.43) hold. It follows from (4.20c), (5.13), (5.14) and $(5.43)_1$ that (5.36) are necessary and sufficient conditions for neutral loading in the motion ${}_2\boldsymbol{\chi}^*$. Then ${}^p_2\dot{\boldsymbol{E}}^*$ and ${}_2\dot{\kappa}^*$ are given by (5.29). Also, (5.25b, c) vanish.

Case XVI: (L, L)

The relations (5.43) hold. In view of $(5.43)_1$, (4.20d), (5.13) and (5.14), necessary and sufficient conditions for loading in the motion ${}_2\boldsymbol{\chi}^*$ are given by (5.37). ${}^p_2\dot{\boldsymbol{E}}^*$ and ${}_2\dot{\kappa}^*$ are given by (5.26b) and (5.26c), respectively.

This completes our analysis of the sixteen cases.

Next, we return to the restriction (4.49). Employing (5.17)–(5.19), (5.21) and (5.11), we deduce from (4.49) that

$$\left(\frac{\partial\bar{\boldsymbol{S}}}{\partial^p\boldsymbol{E}^*}({}_1\mathcal{U}^*) + \frac{\partial\bar{\boldsymbol{S}}}{\partial\kappa^*}({}_1\mathcal{U}^*)\otimes\mathscr{C}({}_1\mathcal{U}^*)\right)[\Delta\boldsymbol{\rho}] + \left(\Delta\left(\frac{\partial\bar{\boldsymbol{S}}}{\partial^p\boldsymbol{E}^*}\right)\right.$$
$$\left. + \frac{\partial\bar{\boldsymbol{S}}}{\partial\kappa^*}({}_1\mathcal{U}^*)\otimes\Delta\mathscr{C} + \Delta\left(\frac{\partial\bar{\boldsymbol{S}}}{\partial\kappa^*}\right)\otimes\boldsymbol{C}({}_1\mathcal{U}^*)\right)[\boldsymbol{\rho}({}_1\mathcal{U}^*)]$$
$$= -\gamma({}_1\mathcal{U}^*)\Delta\left(\frac{\partial g}{\partial\boldsymbol{E}^*}\right) - \Delta\gamma\frac{\partial g}{\partial\boldsymbol{E}^*}({}_1\mathcal{U}^*) + \boldsymbol{0}(\epsilon^2) \text{ as } \epsilon\to 0, \tag{5.44a}$$

where

$$\gamma({}_2\mathcal{U}^*) - \gamma({}_1\mathcal{U}^*) = \Delta\gamma + 0(\epsilon^2) \text{ as } \epsilon\to 0 \tag{5.44b}$$

with

$$\Delta\gamma = \frac{\partial\gamma}{\partial\boldsymbol{E}^*}({}_1\mathcal{U}^*)\cdot\Delta\boldsymbol{E}^* + \frac{\partial\gamma}{\partial^p\boldsymbol{E}^*}({}_1\mathcal{U}^*)\cdot({}^p_2\boldsymbol{E}^* - {}^p_1\boldsymbol{E}^*) + \frac{\partial\gamma}{\partial\kappa^*}({}_1\mathcal{U}^*)({}_2\kappa^* - {}_1\kappa^*)$$
$$= 0(\epsilon) \text{ as } \epsilon\to 0, \tag{5.44c}$$

Next we obtain an expression for ${}_2\rho^*$. It follows from $(4.32)_1$ that

$${}_2\rho^* = {}_1\rho^*{}_1J^*/{}_2J^* \tag{5.45}$$

and hence by (4.44), $(4.7)_2$, (5.1), (5.4) and (5.5)

$${}_2\rho^* = {}_1\rho^*\{\det(\boldsymbol{H} + \boldsymbol{I})\}^{-1} = {}_1\rho^*(1 - \operatorname{tr}\boldsymbol{H}) + 0(\epsilon^2) \text{ as } \epsilon\to 0. \tag{5.46}$$

Turning now to the traction condition (4.27), we note first that it follows from (4.26), (4.44) and (5.16) that

$$ {}_2\boldsymbol{P}^* = (\boldsymbol{I} + \boldsymbol{H})_1\boldsymbol{P}^* + {}_1\boldsymbol{F}^*\Delta\boldsymbol{S}^* + \mathbf{0}(\epsilon^2) \text{ as } \epsilon \to 0. \tag{5.47} $$

It then follows from (4.27) that

$$ {}_2\boldsymbol{p}^* = (\boldsymbol{I} + \boldsymbol{H})_1\boldsymbol{p}^* + {}_1\boldsymbol{F}^*\Delta\boldsymbol{S}^*{}_0\boldsymbol{n} + \mathbf{0}(\epsilon^2) \text{ as } \epsilon \to 0. \tag{5.48} $$

Paralleling (5.47), the relationship between the Cauchy stress tensors ${}_2\boldsymbol{T}^*$ and ${}_1\boldsymbol{T}^*$ is readily seen to be

$$ {}_2\boldsymbol{T}^* = (1 - \operatorname{tr}\boldsymbol{H})_1\boldsymbol{T}^* + {}_1\boldsymbol{T}^*\boldsymbol{H}^T + \boldsymbol{H}_1\boldsymbol{T}^* + \frac{1}{{}_1J^*}{}_1\boldsymbol{F}^*\Delta\boldsymbol{S}^*({}_1\boldsymbol{F}^*)^T, \tag{5.49} $$

where terms of $\mathbf{0}(\epsilon^2)$ as $\epsilon \to 0$ have been neglected.

We digress momentarily in order to compare (5.49) with the corresponding equation, (4.31), of [7] for the purely elastic case. In [7] the constitutive equation for ${}_1\boldsymbol{T}^*$ is of the form†

$$ {}_1\boldsymbol{T}^* = {}_1\rho^*{}_1\boldsymbol{F}^*\frac{\partial\hat{\epsilon}}{\partial\boldsymbol{E}^*}({}_1\boldsymbol{E}^*)({}_1\boldsymbol{F}^*)^T, \tag{5.50} $$

with a similar equation for ${}_2\boldsymbol{S}^*$. In the elastic region, ${}^p_1\boldsymbol{E}^*$, ${}^p_2\boldsymbol{E}^*$, ${}_1\kappa^*$ and ${}_2\kappa^*$ have constant view of (4.26) and (4.32)$_1$

$$ {}_1\boldsymbol{S}^* = {}_0\rho\frac{\partial\hat{\epsilon}}{\partial\boldsymbol{E}^*}({}_1\boldsymbol{E}^*) \tag{5.51} $$

with a similar equation for ${}_2\boldsymbol{S}^*$. In the elastic region, ${}^p_1\boldsymbol{E}^*$, ${}^p_2\boldsymbol{E}^*$, ${}_1\kappa^*$ and ${}_2\kappa^*$ have constant values and recalling (4.31) we may make the identification

$$ {}_0\rho\frac{\partial\hat{\epsilon}}{\partial\boldsymbol{E}^*}({}_1\boldsymbol{E}^*) = \bar{\boldsymbol{S}}({}_1\mathcal{U}^*), \quad ({}^p_1\boldsymbol{E}^*, {}_1\kappa^* \text{ constant}) \tag{5.52} $$

with a similar equation holding for the motion ${}_2\chi^*$. It follows from (5.52) that

$$ \overline{\mathscr{K}} = \frac{\partial^2\hat{\epsilon}}{\partial\boldsymbol{E}^*\partial\boldsymbol{E}^*}({}_1\boldsymbol{E}^*) = \frac{1}{{}_0\rho}\frac{\partial\bar{\boldsymbol{S}}}{\partial\boldsymbol{E}^*}({}_1\mathcal{U}^*), \quad ({}^p_1\boldsymbol{E}^*, {}_1\kappa^* \text{ constant}), \tag{5.53} $$

where $\overline{\mathscr{K}}$ has been introduced for purposes of comparison with [7]. Now consider *Case I*: (E, E) above. It follows from $(5.27)_{2,3}$ and (5.29) that if a material exhibits elastic behavior over some time interval then

$$ \begin{aligned} {}^p_1\boldsymbol{E}^* &= {}_1\boldsymbol{A}(\boldsymbol{X}), \quad {}_1\kappa^* = {}_1k(\boldsymbol{X}), \\ {}^p_2\boldsymbol{E}^* &= {}_2\boldsymbol{A}(\boldsymbol{X}), \quad {}_1\kappa^* = {}_2k(\boldsymbol{X}), \end{aligned} \tag{5.54} $$

where ${}_1\boldsymbol{A}(\boldsymbol{X})$, ${}_2\boldsymbol{A}(\boldsymbol{X})$, ${}_1k(\boldsymbol{X})$ and ${}_2k(\boldsymbol{X})$ represent initial values of the corresponding variables. If we take

$$ {}_1\boldsymbol{A}(\boldsymbol{X}) = {}_2\boldsymbol{A}(\boldsymbol{X}), \quad {}_1k(\boldsymbol{X}) = {}_2k(\boldsymbol{X}), \tag{5.55} $$

then by (5.54)

$$ {}^p_2\boldsymbol{E}^* - {}^p_1\boldsymbol{E}^* = \mathbf{0}, \quad {}_2\kappa^* - {}_1\kappa^* = 0. \tag{5.56} $$

†In [7] we used the notation $\frac{1}{2}(D\hat{\epsilon} + D^T\hat{\epsilon})$ for $\partial\hat{\epsilon}/\partial\boldsymbol{E}^*$.

When (5.56) hold, it follows from (5.49) that

$$ {}_2\boldsymbol{T}^* = (1 - \operatorname{tr}\boldsymbol{H})_1\boldsymbol{T}^* + {}_1\boldsymbol{T}^*\boldsymbol{H}^T + \boldsymbol{H}_1\boldsymbol{T}^* + \tfrac{1}{2}\,{}_1\rho^*{}_1\boldsymbol{F}^*\overline{\mathscr{K}}[({}_1\boldsymbol{F}^*)^T(\boldsymbol{H}+\boldsymbol{H}^T)_1\boldsymbol{F}^*]({}_1\boldsymbol{F}^*)^T, \tag{5.57} $$

where (5.53), (4.32), (5.16b) and (5.6b) have also been used. The result (5.57) coincides with (4.31) of [7].

It is interesting to note that whenever the eqns (5.56) are satisfied,† (5.49) reduces to an equation of the form (5.57).

Returning now to the general case, it follows from (4.48), (4.44), $(5.1)_1$, (5.4), (5.5) and (5.9a) that

$$ {}_2\boldsymbol{n}^* = {}_1\boldsymbol{n}^* + \tfrac{1}{2}\{{}_1\boldsymbol{n}^*\cdot(\boldsymbol{H}+\boldsymbol{H}^T)_1\boldsymbol{n}^*\}_1\boldsymbol{n}^* - \boldsymbol{H}^T{}_1\boldsymbol{n}^* \tag{5.58} $$

where terms of $\mathbf{0}(\epsilon^2)$ as $\epsilon \to 0$ have been omitted. It then follows from (4.22), (5.49) and (5.58) that to within terms of $\mathbf{0}(\epsilon^2)$ as $\epsilon \to 0$,

$$ {}_2\boldsymbol{t}^* = (1 - \operatorname{tr}\boldsymbol{H} + \tfrac{1}{2}\{{}_1\boldsymbol{n}^*\cdot(\boldsymbol{H}+\boldsymbol{H}^T)_1\boldsymbol{n}^*\})_1\boldsymbol{t}^* + \boldsymbol{H}_1\boldsymbol{t}^* + \frac{1}{{}_1J^*}\,{}_1\boldsymbol{F}^*\Delta\boldsymbol{S}^*({}_1\boldsymbol{F}^*)^T{}_1\boldsymbol{n}^*. \tag{5.59} $$

We will now demonstrate that the eqns (5.26b,c) and (5.49) of the approximate theory are properly invariant under arbitrary superposed rigid motions (2.7). Corresponding to the motions $({}_\alpha\boldsymbol{\chi}^+)^*(\alpha = 1, 2)$, the measure defined in $(5.1)_1$ has a value

$$ \epsilon_1^+ = \sup_{({}_1\boldsymbol{x}^+)^*\in{}_1\mathscr{R}^{+*}} \|\boldsymbol{H}^+(({}_1\boldsymbol{x}^+)^*, t)\|, \tag{5.60} $$

with similar expressions for ϵ_2^+, ϵ_3^+ and ϵ_4^+. It follows from (4.52), (4.74), (5.1) and (5.60) that

$$ \epsilon_1^+ = \epsilon_1. \tag{5.61} $$

Similarly, in view of (4.52), (4.75) and $(4.58)_1$, $\epsilon_2^+ = \epsilon_2$, $\epsilon_3^+ = \epsilon_3$ and $\epsilon_4^+ = \epsilon_4$. Consequently, if $\epsilon^+ = \max\{\epsilon_1^+, \epsilon_2^+, \epsilon_3^+, \epsilon_4^+\}$ then, recalling (5.4)

$$ \epsilon^+ = \epsilon. \tag{5.62} $$

Let

$$ \Delta\boldsymbol{E}^{+*} = \tfrac{1}{2}({}_1\boldsymbol{F}^{+*})^T(\boldsymbol{H}^+ + (\boldsymbol{H}^+)^T)_1\boldsymbol{F}^{+*}. \tag{5.63} $$

It then follows with the help of $(4.55)_1$, (4.74) and (5.6b) that

$$ \Delta\boldsymbol{E}^{+*} = \Delta\boldsymbol{E}^*. \tag{5.64} $$

Associated with the motions $({}_\alpha\boldsymbol{\chi}^+)^*(\alpha = 1, 2)$ we may define quantities Δg^+ and ξ^+ as in (5.10b) and (5.12b). Thus, for example,

$$ \begin{aligned} \Delta g^+ = {} & \frac{\partial g}{\partial \boldsymbol{E}^{+*}}({}_1\mathscr{U}^{+*})\cdot\Delta\boldsymbol{E}^{+*} + \frac{\partial g}{\partial {}^p\boldsymbol{E}^{+*}}({}_1\mathscr{U}^{+*})\cdot({}^p_2\boldsymbol{E}^{+*} - {}^p_1\boldsymbol{E}^{+*}) \\ & + \frac{\partial g}{\partial \kappa^{+*}}({}_1\mathscr{U}^{+*})({}_2\kappa^{+*} - {}_1\kappa^{+*}). \end{aligned} \tag{5.65} $$

It is readily seen, with the help of $(4.58)_1$ and (5.64) that each term in (5.65) coincides with

†Even during loading, for example.

the corresponding term in (5.10b), and consequently

$$\Delta g^{+} = \Delta g. \tag{5.66}$$

Similarly, in view of $(4.58)_1$, $(4.56)_1$, (4.75) and (5.12) it easily follows that

$$\xi^{+} = \xi. \tag{5.67}$$

Corresponding to the motions $({}_\alpha\boldsymbol{\chi}^{+})^*(\alpha = 1, 2)$, instead of (5.26b), we have

$${}^{p}_{2}\dot{\boldsymbol{E}}^{+*} = {}^{p}_{1}\dot{\boldsymbol{E}}^{+*} + \{\lambda({}_1\mathscr{U}^{+*})\Delta\rho^{+} + \rho({}_1\mathscr{U}^{+*})\Delta\lambda^{+}\}{}_1\hat{g}^{+*} + \lambda({}_1\mathscr{U}^{+*})\rho({}_1\mathscr{U}^{+*})\xi^{+} \tag{5.68}$$

where

$${}_1\hat{g}^{+*} = \frac{\partial g}{\partial \boldsymbol{E}^{+*}}({}_1\mathscr{U}^{+*})\cdot{}_1\dot{\boldsymbol{E}}^{+*} \tag{5.69}$$

and $\Delta\rho^{+}$ and $\Delta\lambda$ are defined in a manner paralleling (5.19b) and (5.20b), respectively. It follows from $(4.58)_1$, $(4.55)_6$, $(4.56)_3$, (4.19) and (5.69) that

$${}_1\hat{g}^{+*} = {}_1\hat{g}^{*}. \tag{5.70}$$

With the help of $(4.58)_1$, $(4.55)_1$, (4.74), (5.62), (5.67) and (5.70) it is evident that (5.68) coincides with (5.26b). Therefore (5.26b) is a properly invariant statement. A parallel argument demonstrates that (5.26c) is also a properly invariant statement.† Furthermore, it is now clear that the equations that arise in the consideration of the sixteen separate cases are all properly invariant. Likewise, (5.44a) is a properly invariant statement.

The stress ${}_2\boldsymbol{T}^{+*}$ in the motion $({}_2\boldsymbol{\chi}^{+})^*$ is given in accordance with (5.49) by

$${}_2\boldsymbol{T}^{+*} = (1 - \operatorname{tr}\boldsymbol{H}^{+}){}_1\boldsymbol{T}^{+*} + {}_1\boldsymbol{T}^{+*}(\boldsymbol{H}^{+})^{T} + \boldsymbol{H}^{+}{}_1\boldsymbol{T}^{+*} + \frac{1}{{}_1J^{+*}}{}_1\boldsymbol{F}^{+*}\Delta\boldsymbol{S}^{+*}({}_1\boldsymbol{F}^{+*})^{T}, \tag{5.71}$$

where terms of $\mathbf{0}((\epsilon^{+})^2)$ as $\epsilon^{+}\to 0$ have been omitted, and where,

$$\begin{aligned}\Delta\boldsymbol{S}^{+*} &= \frac{\partial\bar{\boldsymbol{S}}}{\partial\boldsymbol{E}^{+*}}({}_1\mathscr{U}^{+*})[\Delta\boldsymbol{E}^{+*}] + \frac{\partial\bar{\boldsymbol{S}}}{\partial^{p}\boldsymbol{E}^{+*}}({}_1\mathscr{U}^{+*})[{}^{p}_{2}\boldsymbol{E}^{+*} - {}^{p}_{1}\boldsymbol{E}^{+*}]\\ &\quad + \frac{\partial\bar{\boldsymbol{S}}}{\partial\kappa^{+*}}({}_1\mathscr{U}^{+*})({}_2\kappa^{+*} - {}_1\kappa^{+*}).\end{aligned} \tag{5.72}$$

It follows from (5.72), (4.31), $(4.58)_1$, $(4.61)_5$, (5.64) and (5.16b) that

$$\Delta\boldsymbol{S}^{+*} = \Delta\boldsymbol{S}^{*}. \tag{5.73}$$

Finally, from (5.49), (5.71), (4.74), $(4.55)_1$, $(4.59)_1$, (5.73) and (4.60) with $\alpha = 1$, we obtain

$${}_2\boldsymbol{T}^{+*} = {}_2\boldsymbol{T}^{*} \tag{5.74}$$

so that the transformation law (4.60), with $\alpha = 2$, is satisfied when ${}_2\boldsymbol{T}^{*}$ is given by (5.49). Thus, (5.49) is a properly invariant statement.

In order to complete the infinitesimal theory of motions superposed on a given motion, we must establish an approximation to the balance of linear momentum (4.34) for the motion ${}_2\boldsymbol{\chi}^{*}$. To this end, in addition to the measures ϵ_1, ϵ_2, ϵ_3, ϵ_4 defined at the beginning

†Equations (5.25b,c) are, of course, also properly invariant.

of this section we need to introduce

$$\begin{aligned}\epsilon_5 &= \sup_{{}_1x^*\in{}_1\mathscr{R}^*}\left\|\frac{\partial H}{\partial_1 x^*}\right\|,\\ \epsilon_6 &= \sup_{{}_1x^*\in{}_1\mathscr{R}^*,\ {}_2x^*\in{}_2\mathscr{R}^*}\left\|\frac{\partial({}_2^{\xi}E^* - {}_1^{\xi}E^*)}{\partial_1 x^*}\right\|,\\ \epsilon_7 &= \sup_{{}_1x^*\in{}_1\mathscr{R}^*,\ {}_2x^*\in{}_2\mathscr{R}^*}\left\|\frac{\partial({}_2\kappa^* - {}_1\kappa^*)}{\partial_1 x^*}\right\|,\\ \epsilon_8 &= \sup_{{}_1x^*\in{}_1\mathscr{R}^*}\left\|\frac{\partial H}{\partial t^*}({}_1x^*, t^*)\right\|,\\ \epsilon_9 &= \sup_{{}_1x^*\in{}_1\mathscr{R}^*}\left\|\frac{\partial \chi^{*\prime}}{\partial t^*}({}_1x^*, t^*)\right\|,\\ \epsilon_{10} &= \sup_{{}_1x^*\in{}_1\mathscr{R}^*}\left\|\frac{\partial^2 \chi^{*\prime}}{\partial (t^*)^2}({}_1x^*, t^*)\right\|,\end{aligned} \tag{5.75a}$$

where in $(5.75\text{a})_{1,2}$ we may define the norm $\|A\|$ of a third order tensor $A = A_{ijk}e_i \otimes e_j \otimes e_k$ by $\|A\|^2 = A_{ijk}A_{ijk}$. Now let

$$\bar{\epsilon} = \max\{\epsilon, \epsilon_5, \epsilon_6, \epsilon_7, \epsilon_8, \epsilon_9, \epsilon_{10}\}. \tag{5.75b}$$

It then follows from (5.49), (4.44), (5.1), (5.4), (5.9a) and (5.75) that

$$\begin{aligned}\frac{\partial_2 T^*}{\partial_2 x^*} &= \frac{\partial}{\partial_1 x^*}\left\{(1 - \operatorname{tr} H)_1 T^* + {}_1T^*H^T + H_1T^* + \frac{1}{{}_1J^*}{}_1F^*\Delta S^*({}_1F^*)^T + \mathbf{0}(\epsilon^2) \text{ as } \epsilon \to 0\right\}\\ &\quad \times \{(I - H + \mathbf{0}(\epsilon^2)\} \text{ as } \epsilon \to 0\\ &= \frac{\partial_1 T^*}{\partial_1 x^*} - \frac{\partial_1 T^*}{\partial_1 x^*}H + \frac{\partial}{\partial_1 x^*}\left\{-(\operatorname{tr} H)_1 T^* + {}_1T^*H^T + H_1T^*\right.\\ &\quad \left. + \frac{1}{{}_1J^*}{}_1F^*\Delta S^*({}_1F^*)^T\right\} + \mathbf{0}(\bar{\epsilon}^2) \text{ as } \bar{\epsilon} \to 0,\end{aligned} \tag{5.76a}$$

where $(\partial_1 T^*/\partial_1 x^*)H$ has a component form

$$\frac{\partial_1 T^*}{\partial_1 x^*}H = \frac{\partial_1 T^*_{ij}}{\partial_1 x^*_m}H_{mk}e_i \otimes e_j \otimes e_k. \tag{5.76b}$$

It follows from (5.76) and (5.49) that

$$\frac{\partial_2 T^*}{\partial_2 x^*} = \frac{\partial_2 T^*}{\partial_1 x^*} - \frac{\partial_1 T^*}{\partial_1 x^*}H + \mathbf{0}(\bar{\epsilon}^2) \text{ as } \bar{\epsilon} \to 0. \tag{5.77}$$

Hence,

$${}_2\text{div}^*\,{}_2T^* = \frac{\partial_2 T^*}{\partial_2 x^*}[I] = {}_1\text{div}^*\,{}_2T^* - \frac{\partial_1 T^*}{\partial_1 x^*}[H^T] + \mathbf{0}(\bar{\epsilon}^2) \text{ as } \bar{\epsilon} \to 0. \tag{5.78a}$$

From (5.76a) we obtain

$${}_2\text{div}^*\,{}_2T^* - {}_1\text{div}^*\,{}_1T^* = \beta + \mathbf{0}(\bar{\epsilon}^2) \text{ as } \bar{\epsilon} \to 0, \tag{5.78b}$$

where

$$\begin{aligned}\beta &= -\frac{\partial_1 T^*}{\partial_1 x^*}[H^T] + {}_1\text{div}^*\left\{-(\operatorname{tr} H)_1 T^* + {}_1T^*H^T + H_1T^* + \frac{1}{{}_1J^*}{}_1F^*\Delta S^*({}_1F^*)^T\right\}\\ &= \mathbf{0}(\bar{\epsilon}) \text{ as } \bar{\epsilon} \to 0.\end{aligned} \tag{5.78c}$$

We also observe that, in view of (5.49),

$$
\begin{aligned}
{}_1\mathrm{div}^*({}_2\boldsymbol{T}^* - {}_1\boldsymbol{T}^*) &= \frac{\partial_1 \boldsymbol{T}^*}{\partial_1 \boldsymbol{x}^*}[\boldsymbol{H}^T] + \boldsymbol{\beta} + \mathbf{0}(\bar{\epsilon}^2) \text{ as } \bar{\epsilon} \to 0 \\
&= \mathbf{0}(\bar{\epsilon}) \text{ as } \bar{\epsilon} \to 0.
\end{aligned} \tag{5.78d}
$$

We note that each term in (5.76), (5.77) and (5.78) is unaltered under superposed rigid motions (2.7).

Next we derive expressions for velocity and acceleration. In view of (4.33), $(4.4)_2$, (4.9) and (4.44)

$$
{}_2\boldsymbol{v}^* - {}_1\boldsymbol{v}^* = \frac{\partial \boldsymbol{\chi}^{*\prime}}{\partial t^*}({}_1\boldsymbol{x}^*, t^*) + \boldsymbol{H}{}_1\boldsymbol{v}^* = \Delta \boldsymbol{v}^* \text{ (say)} \tag{5.79}
$$

and hence

$$
\begin{aligned}
{}_2\dot{\boldsymbol{v}}^* - {}_1\dot{\boldsymbol{v}}^* &= \frac{\partial^2 \boldsymbol{\chi}^{*\prime}}{\partial (t^*)^2}({}_1\boldsymbol{x}^*, t^*) + \boldsymbol{H}{}_1\dot{\boldsymbol{v}}^* + \left(2\frac{\partial \boldsymbol{H}}{\partial t}({}_1\boldsymbol{x}^*, t^*) + \frac{\partial \boldsymbol{H}}{\partial_1 \boldsymbol{x}^*}{}_1\boldsymbol{v}^*\right){}_1\boldsymbol{v}^* \\
&= \Delta \dot{\boldsymbol{v}}^* = \dot{\overline{\Delta \boldsymbol{v}^*}},
\end{aligned} \tag{5.80}
$$

where use has been made of the relations

$$
\begin{aligned}
\dot{\boldsymbol{H}} &= \frac{\partial \boldsymbol{H}}{\partial t^*}({}_1\boldsymbol{x}^*, t^*) + \frac{\partial \boldsymbol{H}}{\partial_1 \boldsymbol{x}^*}({}_1\boldsymbol{x}^*, t^*){}_1\boldsymbol{v}^*, \\
\frac{\partial}{\partial_1 \boldsymbol{x}^*}\left(\frac{\partial \boldsymbol{\chi}^{*\prime}}{\partial t^*}({}_1\boldsymbol{x}^*, t^*)\right) &= \frac{\partial \boldsymbol{H}}{\partial t^*}({}_1\boldsymbol{x}^*, t^*).
\end{aligned} \tag{5.81}
$$

We note that each term in (5.79) and (5.80) is unaltered under superposed rigid motions (2.7). It follows from (5.79), (5.80), $(5.1)_1$; (5.74) and (5.75) that

$$
\begin{aligned}
\Delta \boldsymbol{v}^* &= \mathbf{0}(\bar{\epsilon}) \text{ as } \bar{\epsilon} \to 0, \\
\Delta \dot{\boldsymbol{v}}^* &= \mathbf{0}(\hat{\epsilon}) \text{ as } \bar{\epsilon} \to 0.
\end{aligned} \tag{5.82}
$$

We observe that instead of using the measure ϵ_2 in $(5.1)_2$, we could in view of $(5.81)_1$ have used the maximum of ϵ_5 and ϵ_8 in (5.75a).

If we now substitute (5.78b), (5.46) and (5.80) in the balance of linear momentum (4.34) for the two motions ${}_1\boldsymbol{\chi}^*$ and ${}_2\boldsymbol{\chi}^*$, we find that the difference in body forces is given by

$$
\begin{aligned}
{}_2\boldsymbol{b}^* - {}_1\boldsymbol{b}^* &= -\frac{1}{{}_1\rho^*}\boldsymbol{\beta} + \Delta \dot{\boldsymbol{v}}^* + ({}_1\boldsymbol{b}^* - {}_1\dot{\boldsymbol{v}}^*)\,\mathrm{tr}\,\boldsymbol{H} + \mathbf{0}(\bar{\epsilon}^2) \text{ as } \bar{\epsilon} \to 0 \\
&= \mathbf{0}(\bar{\epsilon}) \text{ as } \bar{\epsilon} \to 0.
\end{aligned} \tag{5.83}
$$

We note that each term in (5.83) is unaltered under superposed rigid motions (2.7).

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